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THE INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS.*

By DAVID EUGENE SMITH, President of the American Commission.

So much has been said and written of late concerning the work of the International Commission on the Teaching of Mathematics, and so closely connected are several members of this Association and this Society with the movement, that I find myself quite at a loss in attempting to impart any new information as to the inception of the work and the general purposes in view. Nevertheless a brief resumé of the organization of the Commission may not be out of place, to be followed by a statement of the problems that particularly confront us here in America.

At the Fourth International Congress of Mathematicians, held at Rome in 1908, the following resolution was adopted: "The Congress recognizing the importance of a comparative examination of the methods and plans of study of the instruction in mathematics in the secondary schools of the different nations, empowers Messrs. Klein, Greenhill, and Fehr to form an International Commission, to study these questions and present a general report to the next Congress."

It was by no means a simple matter to effect this preliminary organization. The general idea was popular — if anything, too popular, and the plan seemed for a time on the point of failing because of this very fact. The difficulty lay in the selection of the organizing committee, the naming of which on the basis of the countries represented would have resulted in an unwieldy body, and one that would naturally have been selected too hurriedly for the best results. In the closing moments of the session of of Section IV, however, there seemed to be a general consensus of opinion that three men were especially fitted to undertake the work of organization,— Professor Klein, because of his high mathematical attainments and his great interest in education; Professor Sir George Greenhill, because of his eminence in the field of applied mathematics and because he represented

*An address delivered before Section A of the American Association for the Advancement of Science, and The American Mathemathical Society, in joint session, at Boston, December 29, 1909.

the country in which the next Congress is to be held; and Professor H. Fehr, because of his position as editor of *L'Enseignement Mathêmatique.* It was expected that these gentlemen would constitute, in the order named, the President, Vice President, and General Secretary of the International Commission which they were empowered to organize, and this expectation has since been realized. Their selection was unanimously approved by the Congress in general session, together with the resolution already mentioned.

These gentlemen set about, in due time, to select representatives from the various countries usually participating in these congresses, consulting with members of learned societies, I am told, and during the year now closing the organization has practically been effected. It would not be appropriate at this time to give the complete list of commissioners chosen, but the names of a few who are acting in this capacity, or are closely connected with the movement as members of important national committees, or otherwise, will serve to show that the work is to be performed in no perfunctory fashion.

Among those who have been chosen to carry on the investigation in Germany are Klein, Staeckel, Gutzmer, Treutlein, Schotten, and Shimmack; in France, Saint-Germain, Bourlet, Appell, Borel, Tannery, Andrê, and Laisant; in Italy, Castelnuovo, Enriques, and Scorza; in Austria, Czuber, Suppantschitsch, and Wirtinger; in Hungary, Beke, Rados, Rátz, and Goldziher; in Great Britain, Greenhill, Godfrey, Fletcher; in Russia, Sonin, Kojalovitch, Vogt; in Denmark, Heegaand, Juel, Trier; in Switzerland, Fehr, Graf, Moser,—all these being men that stand among the leaders in their lines. In the United States, there have accepted positions upon the Advisory Council the president and all of the living ex-presidents of the American Mathematical Society, the president of the American Federation of Teachers of the Mathematical and Natural Sciences, the presidents of Harvard University, Columbia University, and the University of Chicago, and the United States Commissioner of Education. In our country are encluded in the membership of the various committees and sub-committees many of our best known mathematicians and teachers of mathematics, and here, as abroad, the best of spirit has been shown towards the work.

The question that naturally arises in considering such a movement is the very ancient one, *Cui bono?* It is a perfectly proper question and it serves the more clearly to fix in the minds of those charged with responsibility, the definite purpose of the investigation.

Before attempting a brief reply, it may be proper to state what is not the purpose, to the end that the real aims may stand forth the more clearly. It is not the purpose of the Commission, nor of any of its members, so far as appears, to pose as a body of reformers. Nothing has yet developed to show that anyone connected with the movement believes that mathematics is more poorly taught than any other subject, or is in more need of reform. No revolution has been advocated, nor has any one expressed any desire to

have the teaching of mathematics remain stagnant. All articles that have thus far been written in connection with the work show that the members of the Commission desire the same substantial progress in teaching mathematics that is generally found in all other subjects, and no one has taken the attitude of opposition to all that exists, nor has anyone appeared with the dictum that all that is, is bad and that the new and untried is always good. This general attitude of the Commission, if we may judge by what has thus far been done, will argue for lack of progress with some earnest workers, but it is probable that it will meet with the approval of the substantial leaders in mathematics and in education today.

In the second place, it is evidently not the purpose of the Commission to make any direct attempt at uniformity in the various countries. The very manner of conducting the investigation shows that each country has gone to work at the problem in the way that seems to it best, without the slightest regard to the methods employed in other countries; and this independence, this recognition of the peculiar needs, aims, and means at the disposal of the various countries, will undoubtedly continue to be seen as the work progresses. It is not, therefore, proposed to revolutionize the teaching of mathematics, nor is it proposed to attempt to bring all nations to one point of view as to schools, as to teaching, as to subjects taught, or as to curricula.

From the time of the Greek mathematicians until now, it has been recognized that a negative statement is never satisfactory, and so it is proper to proceed to a positive statement of some of the reasons for this word-wide investigation.

The prime reason is unquestionably to be found in the value of an interchange of views and experiences among nations. Why should the American Association for the Advancement of Science, or the American Mathematical Society, or the American Federation of Teachers of the Mathematical and Natural Sciences exist? Is it not enough that we have the faculty meetings, and local reading circles, and even state organizations? Have we not journals and reports and text-books and treatises to keep us mentally alert? Then why do we need such national gatherings as the present one, and why should we give up our well-earned holidays to attend them? These questions seem puerile when applied to our present gathering, for each of us is conscious that he gains much inspiration from such a meeting, and that he returns to his work with a broader horizon because he has been conducted to other points of view. He sees the national problem instead of his narrow local one, and his work is the stronger because of this fact. You cannot weigh this value, nor is it to be measured in volume or in depth, for it is incommensurable with any units of the laboratory or of trade,—but our sixth sense knows it, and it is even more real than it would be if expressed in the mere figures of a scale.

And if the village point of view is narrow, and even that of the city,

and of the state, and of this section or that of a country like ours, so too is the national point of view narrow as compared with that of the world of which it must be, whether it will or no, a part. This, then, is the great *raison d'être* of the Commission,—to act as a clearing house, to let each part of the world see what the other part is doing in the teaching of mathematics. Uniformity is no more desirable nor attainable than uniformity in dress, in government, or in speech. The movement will tend in this direction, just as all intercourse tends to uniformity in these other lines, but centuries will elapse before any of these results will be reached, if unfortunately they are ever attained.

Such a movement, looking to telling the rest of the world what any specific nation is doing in the teaching of mathematics, carries with it the necessity, on the part of that nation, of taking stock of its own results, and of indulging in a very healthy form of introspection. It is most interesting to those of us who are intimately connected with the investigation, to see how much we all have to learn of the courses of study, the amount that can be well done in a year, and the topics that are treated in the various parts of our country. I will illustrate by two simple but typical questions: (1) What are the aims, and consequently what should be the nature of the work, in mathematics, in the large city commercial high schools of this country? (2) What are the requirements in mathematics for the doctor's degree in say three universities belonging to the Association of American Universities? It would seem that these questions ought to be easily answered by several persons in an audience like this, but we find difficulty in discovering anyone to answer them, in a much larger circle than the one here represented. A second large reason, then, for this investigation is that we may know not only what the world at large is doing, but also our own aims and curricula and efforts at improvement.

But more specfically, what is it that we propose to investigate?—that the rest of the world wishes to know? that we ourselves need also to inquire into?

In the first place, it was at once seen that the limitation to secondary schools was meaningless. The term signifies one thing in America, but something radically different in Germany, and something different still in other countries. Even with us there is no well-marked boundary between the elementary school with six, seven, eight, or nine years, and the secondary school with six, five, or four years, or even less. It was therefore early decided to extend the investigation so that it might include the entire field of mathematics required for any purpose, from the kindergarten through the college, the technical school, and the professional school. Besides this, since the investigation includes the preparation of teachers, it must also include a statement of the general nature of the graduate work in mathematics in the universities, and in particular of the requirements for the various advanced degrees in universities of recognized standing.

With the general scope of the work thus set forth, I pass to a consideration of a few of the large topics to be investigated in this country.

The first of these topics is the mathematics of the general elementary school, where a few years ago the obsolete business problem and the remains of the ancient theory of numbers were looked upon as of paramount importance, but where now the needs of modern daily life make up the bulk of the work. The change here has been greater than in any other part of our American education in mathematics, save only in the graduate university work. To report upon this change, and to set forth the present status of the work, for the various types of schools, is a labor of no little difficulty, and the world at large will look upon it as one of our most important contributions.

There is next considered certain special kinds of elementary schools, of which we hear and know little because they are still so new. The mathematics of the trade school is not that of the public school, nor is that of the large corporation industrial schools, now springing up in various parts of the country, like that of the trade school. We have not yet differentiated in this field as they have, for example, in Switzerland, and perhaps we shall never need to do so, but it is certain that these schools are with us, and that others are coming, and that their needs in mathematics, and their ways of meeting these needs, will be of interest to all who are engaged in the furthering of education.

One of the largest problems that confronts us is the study of the public general secondary schools of the country, partly because it is here that mathematics, beyond mere computation, may be said to begin, and partly because there is more criticism of this work than of any other save in the first two years of college. Fortunately we have in Mr. Evans of this city an organizer of proved ability, and with him we have a committee of very earnest workers. Anyone who is at all familiar with these public schools knows that there is much divergence in the courses in mathematics in different parts of the country, and in schools intended for boys, for girls, or for both. It is also apparent that the new movement towards a six-year elementary school, followed by a six-year high school, will considerably modify the mathematics of the latter. With six years in the high school, the present mathematics will begin earlier, it may possibly be taken more leisurely, and eventually it may well lead into higher lines of work. The problem, therefore, assumes an interest that a few years ago would not have been expected. Add to this the numerous efforts made at present to frame new syllabi in subjects like algebra and geometry, and it is evident that the Committee has a labor of no small extent to perform.

Then there is another problem that is quite different, namely, that of the private secondary school, whether for boys alone, for girls alone, or for both together. These are generally fitting schools for college. Instead of a small per cent. going to college, as in the public schools, the rest going

into business, here these per cents are reversed, and the nature of the work in mathematics is, or may be, very different. As a matter of fact, it is different, and more different than one would at first expect.

An important question, and one that is by no means settled, is that which relates to the mathematics in the normal schools, public and private. Upon this depends the equipment of the teachers who are to instruct the new generation of elementary pupils in the early principles of mathematics, largely determining whether or not they will care for the subject in later years. Of late there has been a tendency to weaken the mathemathics in these schools, and it is hoped that this report will determine the needs and perhaps the future policy of such institutions in this important field.

There is arising in our country at present a type of secondary school that is well known on the continent of Europe but that has yet to make its way with us. This is the technical secondary school. We have, for example, even in the present year a notable development of the agricultural high school, of the commercial high school, and of the general technical or manual training high school. What shall the mathematics be in these institutions? Shall it be Euclid revised and algebra through quadratics, as in the ordinary high school? Or shall it be mathematics correlating directly with the peculiar work of the type of school in question?

Related to these schools is the rapidly increasing number of evening technical schools, of private correspondence schools, of schools for licensed accountants, and of industrial schools for negroes and Indians. Some of these institutions are teaching mathematics even through applied calculus, and are doing it with a spirit that would put the average college class to shame, and it will help us all to know their work more fully.

One of the most important problems that we have relates to the examination question. We commonly look at England as examination-ridden, and in the East we often hear it said that the West and Middle West teach mathematics so loosely that they dare not submit their pupils to the Eastern tests. Here, therefore, are these types,—the English, the extreme of the examination system; the West, the extreme of the non-examination system; and the East occupying a middle ground. It is well that we endeavor fairly to weigh the advantages of these various plans, to consider the nature of the promotion in the elementary grades, to investigate the results of the several plans for admission to college, and to inform ourselves as to the examinations in mathematics set by the state and local authorities for those desiring to teach in the public schools. If we do not, as a result of all this work, attain to uniformity, at least we shall be more tolerant of the plans of others, if not of our own.

America has of late branched out geographically for good or for ill, and it is expected that we shall hear, as part of this report, of the work being done in mathemathics in the Philippines, in Hawaii, in Porto Rico, and in Alaska. Nor will this be merely a discussion of the elementary schools,

for Hawaii has more than one good college, and Manila has a university so old that most of ours seem young in comparison.

We have in this country certain influences that tend to improve the work of the teacher that will be of interest to other countries, and about which we ourselves may profitably be better informed. Our scientific societies and periodicals are stimuli to our scientific activity, and are characterized by a generosity of spirit that will be the envy of several countries. The nature of these organizations and publications, including the American Association for the Advancement of Sciences, and our various mathematical journals, will be the subject of one of our important reports. We are favored, too, in having so many associations of teachers of mathematics, and to them we owe most of the syllabi in algebra and geometry that are now appearing. The teachers' institute, as conducted in our various states, seems to be indigenous to our country, and while it has not in general done much for mathematics, it has, in the high school sections, often proved of service. These and similar questions will be discussed by a committee appointed for this special purpose.

There has grown up in this country a line of technological work which we have reason to regard with much satisfaction,—that of collegiate grade, represented by institutions like the Massachusetts Institute of Technology. Such schools are not unique with this country; indeed, we borrowed the idea from Germany, Switzerland, France, and Belgium. But they have steadily increased in number, in dignity, and in their ability to offer strong courses in engineering mathematics. But what this work is, and how it differs in the technological schools from the work in the technological departments of colleges and universities, are matters upon which we should all welcome exact information.

Allied to this problem is that of the mathematics taught in other professional schools, such as those for the training of army and navy officers, and particularly in the higher institutions for the training of teachers that are beginning to be established in this country. The question of how we proceed to train a teacher of mathematics for the secondary school and college, in the highest type of college for this purpose, will be of interest abroad and not without its value at home.

From the educational standpoint the mathematical work of the college is commonly felt, in this country, to be the least progressive. Perhaps it should be so, but at any rate it will be of service to us to know exactly what is expected of the freshman year, and of the sophomore year, in an average American college. Whatever exact information we can get as to the aims, the general plan of presentation, and the results, of these two years, will be stimulating in all institutions of this type. Perhaps as important a question as we have to consider is that relating to the nature of the mathematics in the women's colleges, compared with that in the colleges for men. Is it the same? Is the ability the same? Such are some of the questions that the world of mathematics would like to have scientifically considered.

Perhaps the most interesting question of all, from the standpoint of mathematics, is that relating to the graduate work done in the universities of this country,—that is, in the few institutions of high grade that are generally accepted as prepared for genuine work of this character. One of our committees has this problem in hand, one of its sub-committees having the question of the courses of instruction, another of the preparation for research and the doctor's degree, and another of the preparation of instructors for colleges and universities. The report of this committee may well be expected to set a standard of excellence that shall be recognized for many years to come, in this country.

And finally, there are certain general questions that need to be considered, particularly for the information of other countries. One of these has to do with a schematic survey of American educational institutions, their sequence and inter-relations. The fact that this particular topic is under the direction of a sub-committee headed by the Commissioner of Education of the State of Massachusetts, is a guarantee that the work will be well done. The second of these general topics has to do with the scope and arrangement of the mathematical curriculum as a whole, as distinguished from the arrangement of limited portions as considered by various other committees.

In America, as in some other countries, the work is carried on by means of committees. Certain other countries are conducting the investigation through individuals. In this country we have fifteen committees, sixty-one sub-committees, and a total of about two hundred and seventy members. These committees and sub-committees are working out their problems in their own way, some by means of elaborate questionnaires, others through associations, others from statistical reports, and others through investigations previously made by their members. It is hoped that a considerable number of the sub-committee reports will be ready in February, 1910, and that all of the committee reports will be ready the following May. Some extensions must be allowed, but it is hoped that these will not delay the ultimate stages of the work. The American report must be ready a year hence, and the great labor that must be expended upon it leads to the hope that all who receive questionnaires will respond heartily and speedily to the end that the burden may not be made any heavier.

Such is the general purpose and such is the general plan of this work. It imposes upon all of us who are connected with it a very heavy burden. But the work must be done for the reputation of the country and for the sake of its effect both here and abroad. It is to the sympathetic interest of societies like those that I address that we look for our chief encouragement and assistance, and it is upon teachers and mathematicians like yourselves that we have called to fill the various committees and to work with us, *con amore* as all of us must, to make the labor result in success and be of value to ourselves and to our confères in other parts of the world.

NOTE ON MAXIMA AND MINIMA BY ALGEBRAIC METHODS.

By N. J. LENNES, Massachusetts Institute of Technology.

The pedagogical interest of this note seems sufficient to warrant its publication. The problems on maxima and minima in the Calculus are admittedly of considerable value in that they furnish interesting and concrete applications of the theory of derivatives, and also because of the importance of the problems themselves. Since nearly all problems of this class, which involve algebraic functions only and which are readily soluable by finding values which make the first derivatives vanish, may also be solved very easily by elementary algebraic methods, it would seem desirable to do so. And all the more so since the algebraic method affords an excellent illustration of the use of the graph and besides brings into play the relations between the roots and the coefficients of an algebraic equation. To those who never reach the calculus but who do study college algebra it may be worth while, for the sake of these problems themselves, to solve them rather than an equal number of abstract exercises.

If the expression whose maximum or minimum we seek is of the second degree we need only to find the highest or lowest point of a curve

$$y=a_0x^2+a_1x+a_2$$

which may be done by solving this equation simultaneously with

$$y=b$$

and then giving b such value that the two roots are identical. That is, the line is made tangent to the curve.

If in a similar manner we seek the intersection points of the cubic

$$y=a_0x^3+a_1x^2+a_2x+a_3$$

and the line

$$y=b,$$

we have an equation yielding three roots. If the line is to be tangent to the curve at one point two of these three roots are coincident. Representing the coincident roots by r_1 and the remaining root by r_2 we have

$$\begin{cases} 2r_1+r_2=\dfrac{a_1}{a_0} \\ 2r_1 r_2+r_1^2=\dfrac{a_2}{a_0} \end{cases}.$$

Of the two values of r_1 obtained by solving this set of equations, one clearly corresponds to a maximum and the other to a minimum. Indeed, this is at once evident to the eye by constructing a graph of the cubic.

In case a_0 is positive the maximum of the function $a_0x^3+a_1x^2+a_2x+a_3$ is to the *left* of its minimum, and hence the smaller value of r_1 obtained above corresponds to a maximum and the greater value to a minimum.

If $r_1=r_2$ or if these are both complex there is, of course, neither maximum nor minimum.

Clearly, the method applies to any curve whose equation is reducible to the form

$$x^3+a_1x^2+a_2x+f(y)=0.$$

where a_1 and a_2 are constants and $f(y)$ any real function whatever of y. It may, of course, be impossible to find the value of y at the maximum or minimum point, though the values of x for such point may always be found.

In a similar manner the maxima and minima points of the curve

$$x^n+a_1x^{n-1}+\ldots+a_{n-1}x+f(y)=0$$

may be found by solving an equation of the $(n-1)$st degree. In the case of the biquadratic the details are as follows:

Suppose two of the roots of the equation

$$x^4+a_1x^3+a_2x^2+a_3x+f(y)=b$$

are coincident. Denote the coincident roots by r_1 and the other two roots by r_2 and r_3. Then

$$\begin{cases} 2r_1+r_2+r_3=-a_1 & (1) \\ r_1^2+2r_1r_2+2r_1r_3+r_2r_3=a_2 & (2) \\ r_1^2r_2+r_1^2r_3+2r_1r_2r_3=-a_3 & (3) \end{cases}$$

From (1) and (2),

$$r_1^2+2r_1(-a_1-2r_1)+r_2r_3=a_2 \qquad (4)$$

From (1), (4) and (3),

$$r_1^2(-a_1-2r_1)+2r_1[a_2-r_1^2+2r_1(a_1+2r_1)] = -a_3 \qquad (5)$$

or

$$4r_1^3+3a_1r_1^2+2a_2r_1=-a_3 \qquad (6)$$

The three values of r_1 obtained by solving equation (6) will be the values of x for which the given curve has turning points. The middle value will necessarily belong to a maximum and the other two to minima. In case two of the roots are coincident the remaining root will give a minimum and in case they are all coincident their value gives a minimum.

Again, these statements are at once rendered evident by means of the graph.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

NOTE.—Thomas S. Clarkson remarks that the solution of 321 does not satisfy the conditions of the problem, since the whole number of shares is not to exceed 200. He contends that the problem is impossible as stated, and suggests that it may have been intended that each man's share is not to exceed 200. We think the problem is stated as intended and is therefore impossible. ED. F.

325. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

I have a chronometer whose rate is uniform. When it indicates t_1 time at Washington I find that it is h_1 hours slow. I take it to Philadelphia and when it indicates t_2 time, the local time of Philadelphia is h_2 hours faster. I bring my chronometer back to Washington and find that when it indicates t_3 time it is h_3 hours slow. If $t_1=5$ A. M., $t_2=7$ hours, 54 minutes, $t_3=11$ hours, 46 minutes A. M., $h_1=1$ hour, $h_2=1$ 203/900 hours, $h_3=1$ 7/30 hours, find the difference of longitude between Washington and Philadelphia.

Solution by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill, and the PROPOSER.

$(h_3-h_1)/(t_3-t_1)$=error for one hour.

$(t_2-t_1)(h_3-h_1)/(t_3-t_1)$=error at time t_2 in Philadelphia.

$\therefore h_1+t_2+(t_2-t_1)(h_3-h_1)/(t_3-t_1)$=time at Washington.

h_2+t_2=time at Philadelphia.

$h_2+t_2-h_1-t_2-(t_2-t_1)(h_3-h_1)/(t_3-t_1)$=difference of time between the two cities=T.

$$\therefore T=\frac{h_2(t_3-t_1)+h_3(t_1-t_2)+h_1(t_2-t_3)}{t_3-t_1}.$$

$15T=L$=difference in longitude.

$$\therefore L=\frac{15(h_1t_2+h_2t_3+h_3t_1-h_1t_3-h_2t_1-h_3t_2)}{t_3-t_1}.$$

Putting $h_1=1$ hour, $h_2=1\frac{203}{900}$ hours, $h_3=1\frac{7}{30}$ hours, $t_1=5$ hours, $t_2=$ 7 hours 54 minutes, $t_3=11$ hours 46 minutes, we get

$$L=\tfrac{113}{60}^\circ=1^\circ\ 53'.$$

Also solved by T. S. Clarkson.

326. Proposed by R. D. CARMICHAEL, Princeton University.

Is the series, of which the nth term is $\dfrac{1.3.5.7\ldots(2n-1)}{(n+1)!\,2^n.(2n+3)}$ convergent? If so, find its sum.

Solution by E. B. ESCOTT, Ann Arbor, Mich.

Apply the test given in Hall and Knight's *Higher Algebra*, 3rd ed., p. 244.

If $\underset{n \doteq \infty}{\text{Lim}} \left[n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right] > 1$, the series is convergent. In this case

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)2(2n+5)}{(2n+1)(2n+3)}.$$

The above limit$=\frac{5}{2}$. Therefore, the series is convergent.

To find its value, we may use the following method due to Euler:

$$\text{Let } y = \ldots + \frac{1.3.5\ldots(2n-1)}{(n+1)!\,2^n\,(2n+3)} x^{2n+3} + \frac{1.3.5\ldots(2n+1)}{(n+2)!\,2^{n+1}(2n+5)} x^{2n+5} + \ldots$$

$$\frac{dy}{dx} = \ldots + \frac{1.3.5\ldots(2n-1)}{(n+1)!2^n} x^{2n+2} + \frac{1.3.5\ldots(2n+1)}{(n+2)!2^{n+1}} x^{2n+4} + \ldots$$

$$\frac{d^2y}{dx^2} = \ldots + \frac{1.3.5\ldots(2n-1)}{n!2^{n-1}} x^{2n+1} + \frac{1.3.5\ldots(2n+1)}{(n+1)!2^n} x^{2n+3} + \ldots$$

Divide by x^3 and integrate,

$$\int \frac{1}{x^3} \frac{d^2y}{dx^2} dx = \ldots + \frac{1.3.5\ldots(2n-3)}{n!\;2^{n-1}} x^{2n-1} + \frac{1.3.5\ldots(2n-1)}{(n+1)!2^n} x^{2n+1} + \ldots$$

Multiply by x and integrating again,

$$\int x \int \frac{1}{x^3} \frac{d^2y}{dx^2} dx\,dx = \ldots + \frac{1.3.5\ldots(2n-3)}{n!\;2^{n-1}(2n+1)} x^{2n+1} + \frac{1.3.5\ldots(2n-1)}{(n+1)!\,2^n\,(2n+3)} x^{2n+3}$$

$$+ \ldots = y.$$

Then differentiating, dividing by x, and differentiating again, we have

$$\frac{1}{x^3} \frac{d^2y}{dx^2} = \frac{1}{x} \frac{d^2y}{dx^2} - \frac{1}{x^2} \frac{dy}{dx}.$$

This differential equation may be written

$$(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.$$

Solving the differential equation, we have

$$y=\frac{c}{2}[x\surd(1-x^2)+\sin^{-1}x].$$

To verify that this gives the given series, we have by differentiating,

$$\frac{dy}{dx}=c\surd(1-x^2)=c(1-\frac{1}{2}x^2-\frac{1}{2!\,2^2}x^4-\frac{1.3}{3!\,2^3}x^6-\ldots$$

$$-\frac{1.3.5\ldots(2n-1)}{(n+1)!\,2^{n+1}}x^{2n+2}-\ldots).$$

Integrating, $y=c(x-\frac{1}{2.3}x^3-\frac{1}{2!2^2.5}x^5-\frac{1.3}{3!2^3.7}x^7-\ldots$

$$-\frac{1.3.5\ldots(2n-1)}{(n+1)!\,2^{n+1}(2n+3)}x^{2n+3}-\ldots).$$

Putting $c=-2$, we have

$$\lim_{x\doteq 1}[-(x\surd(1-x^2)+\sin^{-1}x)]=-\frac{\pi}{2}=-2+\frac{1}{3}+\frac{1}{2!\,2.5}+\frac{1.3}{3!2^2.7}+\ldots$$

Also solved by G. B. M. Zerr, who found for the limit of the sum of n terms of the series as $n\doteq\infty$, $\frac{5}{3}-\frac{1}{2}\pi$.

GEOMETRY.

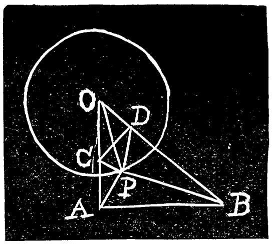
349. Proposed by J. A. CAPARO, Notre Dame University, Notre Dame, Indiana.

Given the radius of a circular smooth cylinder and its position with respect to a source of light and the eye. Find a geometrical construction to determine the line of brilliancy.

Solution by C. N. SCHMALL, New York City.

By a simple examination of the data* it is clear that the problem reduces to the following: Given a circle and two points A, B, outside (in same plane), to find a point P on the circumference, such that AP and

*See *Encyclopedia Britannica*, Vol. XIV, page 589.



BP make equal angles with the radius drawn to *P*.

This is known as Alhazen's Problem and does not admit of a solution by ruler and compasses only. However, by the use of an hyperbola an approximate solution can be effected.

Construction. Let O be the center of the circle, a its radius. Take C and D so that $AO.OC=a^2=BO.OD$.

Now, the locus of the vertices of the triangles whose base is CD and whose base angles have a constant difference $(OCD-ODC)$ is well known to be a hyperbola. This will cut the circle in four points, of which let P be one. This is the point required.

Proof. $\angle CDP-\angle DCP=\angle OCD-\angle ODC$ (by construction). Transposing and adding,

$\therefore \angle OCP=\angle ODP$...(1).

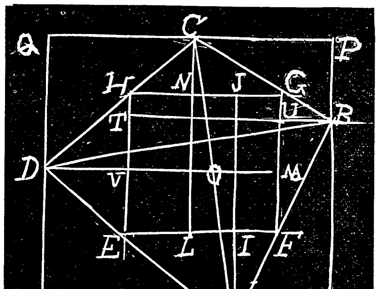
Also in the triangles AOP, POC, we have $AO{:}OP=OP{:}OC$ (by construction).

$\therefore \angle APO=\angle OCP$. Similarly, $\angle BPO=\angle ODP$ by (1).

$\therefore \angle APO=\angle BPO$, and $\therefore \angle APR=\angle BPR$. Q. E. F. Q. E. D.

350. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Given the quadrilateral $AB=a=225$, $BC=b=153$, $CD=c=207$, $DA=d=135$, $AC=e=240$. Find the side of the square inscribed in this quadrilateral having a corner in each side.



Solution by the PROPOSER.

Let $ABCD$ be the given quadrilateral, $EFGH$ the inscribed square, $RPQS$ the circumscribed rectangle having its sides parallel to the sides of the square. Draw AIJ, BUT, CNL, DVM perpendicular, respectively, to EF and HG, FG and HE, HG and EF, HE and GF.

Let $AB=a$, $BC=b$, $CD=c$, $DA=d$, $AC=e$, $BD=f$, O the intersection of AC, BD, $\angle COB=\beta$, $\angle ACL=\theta=\angle CAJ$, $\angle BDM=\phi=\angle DBT$, $\angle BAC=\delta$, $\angle DAC=\gamma$, $\angle BCA=\rho$, $\angle DCA=\mu$, area $ABCD=\triangle$.

Then $\phi=\frac{1}{2}\pi-(\beta-\theta)$, $RB=a\cos(\delta-\theta)$, $RA=a\sin(\delta-\theta)$, $DS=d\cos(\gamma+\theta)$, $AS=d\sin(\gamma+\theta)$, $DQ=c\cos(\mu-\theta)$, $CQ=c\sin(\mu-\theta)$, $BP=b\cos(\rho+\theta)$, $PC=b\sin(\rho+\theta)$, $AI+x+CN=AJ+CN=e\cos\theta$, $BU+x+DV=BT+DV=f\cos\phi$.

$\therefore BT+DV=f\sin(\beta-\theta)$.

$x^2+\frac{1}{2}x(AI+BU+CN+DV)=\triangle=\frac{1}{2}x(AI+x+CN)+\frac{1}{2}x(BU+x+DV)$.

$\therefore x[e\cos\theta+f\sin(\beta-\theta)]=2\triangle$...(1).

$\triangle+\frac{1}{4}[a^2\sin 2(\delta-\theta)+b^2\sin 2(\rho+\theta)+c^2\sin 2(\mu-\theta)+d^2\sin 2(\gamma+\theta)$
$=ef\cos\theta\sin(\beta-\theta)$...(2).

Reducing (2), and remembering that $4\triangle-2ef\sin\beta=0$, we get

$$\tan 2\theta=\frac{a^2\sin 2\delta+b^2\sin 2\rho+c^2\sin 2\mu+d^2\sin 2\gamma-2ef\sin\beta}{a^2\cos 2\delta-b^2\cos 2\rho+c^2\cos 2\mu-d^2\cos 2\gamma-2ef\cos\beta}\ldots(3).$$

When $a=225$, $b=153$, $c=207$, $d=135$, $e=240$. Then $f=277.4$, $\triangle=30656.46$, $\delta=38°\ 14'\ 54''$, $\rho=65°\ 33'\ 40''$, $\gamma=59°\ 24'\ 36''$, $\mu=34°\ 9'\ 22''$, $\beta=67°\ 3'\ 52''$.

$\therefore\tan 2\theta=-.345608=2\tan\theta/(1-\tan^2\theta)$.

$\therefore\tan\theta=5.954833$ or -0.167930.

$\theta=80°\ 28'\ 2''$ or $170°\ 28'\ 2''$ and $x=-2496.97$ or -121.044.

Therefore, two squares can be inscribed in this quadrilateral, the smaller one truly inscribed and the larger with its corners on the sides.

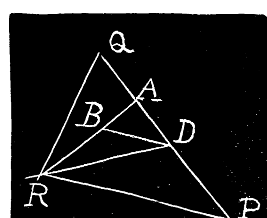
Also solved by J. Scheffer, and V. M. Spunar.

351. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Given an isosceles right triangle with hypothenuse h; an isosceles triangle with two sides h and two angles $A=22°\ 30'$; a right angle triangle with the same angle A and opposite side $h/\sqrt{2}$; a triangle with the same angle A, opposite side h, and an angle 45°. Form a triangle whose four pieces are these four triangles, and prove geometrically that it is isosceles.

Solution by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.

Let ABD be the isosceles right triangle with hypothenuse $BD=h$, sides AB, $AD=h/\sqrt{2}$.



Produce AB to R, making $BR=BD$ and connect RD. Since $\angle ABD=45°$, $\angle RBD=135°$.

$\therefore\angle BRD=\angle RDB=22°\ 30'=A$, and BRD is the isosceles triangle with sides h and opposite angles A. Take Q in DA produced so that $AQ=AD$ and join RQ. Then the right triangle ARQ has angle $ARQ=A$ and $AQ=h/\sqrt{2}$. Produce AD to P making $DP=h$ and join RP. Triangle DRP has angle $DRP=A$, $\angle P=45°$, and side DP opposite $\angle DRP=h$.

Now $\angle PQR=67°\ 30'=3A$, $\angle ARQ=\angle ARD=\angle DRP=A$.

$\therefore\angle QRP=3A$. $\therefore\angle Q=\angle QRP$, and the triangle PQR is isosceles.

CALCULUS.

281. **Proposed by S. A. COREY, Hiteman, Iowa.**

Prove that if n be a positive integer greater than unity,

$$\text{Log}n=(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{2n})-C+\frac{B_1}{2n^2}-\frac{B_2}{4n^4}+\frac{B_3}{6n^6}-\frac{B_4}{8n^8}+\ldots (1).$$

REMARK.—This development may be obtained by employing the formula given by the proposer in *Annals of Mathematics*, second series, Vol. 5, No. 4, July, 1904, but other proofs are also desired. It will be observed that (1) offers a ready method of evaluating C, which is remarkably simple and very rapidly convergent if $n>$ or $=10$. Compare with method given by Mr. Bromwitch in *Messenger of Mathematics*, October, 1906.

I. **Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

In Art. 304, Todhunter's *Integral Calculus*, we find the following result deduced:

$$\Sigma\phi(x)=C+\frac{1}{h}\int\phi(x)dx-\tfrac{1}{2}\phi(x)+B_1\phi'(x)\frac{h}{2}-B_2\phi'''(x)\frac{h^3}{4!}$$

$$+B_3\phi^{\text{v}}(x)\frac{h^5}{6!}-B_4\phi^{\text{vii}}(x)\frac{h^7}{8!}+\ldots$$

Let $h=1$, $\phi(x)=1/n$. Then $\phi'(x)=-1/n^2$, $\phi'''(x)=-3!/n^4$, $\phi^{\text{v}}(x)=-5!/n^6$, $\phi^{\text{vii}}(x)=-7!/n^8$.

$$\Sigma\phi(x)=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n-1}.$$

$\therefore\ \Sigma\left(\dfrac{1}{n}\right)=C+\log n-\dfrac{1}{2n}-\dfrac{B_1}{2n^2}+\dfrac{B_2}{4n^4}-\dfrac{B_3}{6n^6}+\dfrac{B_4}{8n^8}-\ldots$ whence the given result.

II. **Solution by C. N. SCHMALL, New York City.**

Take the well known relation

$$\Gamma(x+2)=(x+1)\Gamma(x+1)\ldots(1)$$

and take logarithms of both sides,

$$\log\Gamma(x+2)=\log\Gamma(x+1)+\log(x+1)\ldots(2).$$

Now define a function $\phi(x)$ by the relation

$$\phi(x)=\frac{d}{dx}\log\Gamma(1+x)\ldots(3).$$

Differentiating equation (2) we have

$$\phi(x+1)=\phi(x)+\frac{1}{x+1}\dots(4).$$

$\therefore \phi(x)=\phi(x+1)-\frac{1}{x+1}$. Again, $\phi(x+2)=\phi[(x+1)+1]$, and by (4),

$$\phi[(x+1)+1]=\phi(x+1)+\frac{1}{(x+1)+1}=\phi(x+1)+\frac{1}{x+2}=\phi(x)+\frac{1}{x+1}+\frac{1}{x+2}$$

Transposing, $\phi(x)=\phi(x+2)-\frac{1}{x+1}-\frac{1}{x+2}$. Hence, by induction, we have

$$\phi(x)=\phi(x+n)-\left[\frac{1}{x+1}+\frac{1}{x+2}+\dots+\frac{1}{x+n}\right]\dots(5),$$

where n is an integer. Put $x=0$, then

$$\phi(n)=-C+1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{n}\dots(6),$$

where $-C=\phi(0)$. Now (6) may be written

$$\phi(n)=-C+\int_0^1(1+z+z^2+\dots+z^{n-1})\,dz=-C+\int_0^1\frac{(1-z^n)}{1-z}dz\dots(7).$$

This defines $\phi(n)$ for integral values of n. Again, by (7),

$$\phi(n+1)-\phi(n)=\int_0^1\frac{(1-z^{n+1})\,dz}{1-z}-\int_0^1\frac{(1-z^n)\,dz}{1-z}=\int_0^1\frac{(z^n-z^{n+1})\,dz}{1-z}$$

$$=\int_0^1\frac{z^n(1-z)}{1-z}dz=\int_0^1 z^n dz=\frac{1}{n+1}.$$

Putting x for n, this becomes

$$\phi(x+1)-\phi(x)=\frac{1}{x+1}$$

which agrees with (4). Hence n may also be a fraction; and we have, in general, from (7),

$$\phi(x)=-C+\int_0^1\frac{(1-z^x)}{1-z}dz=-C+\int_0^1(1-z^x)(1+z+z^2+\ldots)dz$$

$$=-C+\frac{x}{x+1}+\tfrac{1}{2}\frac{x}{x+2}+\tfrac{1}{3}\frac{x}{x+3}+\ldots(8).$$

Replace $\phi(x)$ by its value from (3), and integrate,

$$\log\Gamma(1+x)=-Cx+x-\log(1+x)+\tfrac{1}{2}x-\log(1+\tfrac{1}{2}x)+\tfrac{1}{3}x-\log(1+\tfrac{1}{3}x)+\ldots$$

Raise e to the power of each side,

$$\Gamma(1+x)=e^{-cx}.\frac{e^x}{1+x}.\frac{e^{\frac{1}{2}x}}{1+\frac{1}{2}x}.\frac{e^{\frac{1}{3}x}}{1+\frac{1}{3}x}\ldots(9).$$

Putting $x=1$, $1=e^{-c}.\tfrac{1}{2}e.\frac{2e^{1/2}}{2}.\frac{3e^{1/3}}{4}.\frac{4e^{1/4}}{5}\ldots$ Taking logarithms,

$$0=-C+1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+(1/n)-\log(n+1).$$

$\therefore\log(n+1)=-C+1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+(1/n)\equiv\phi(n)$ [by (6)], when n becomes infinite. Hence, for a very large value of n we have, approximately,

$$\phi(n)=\log(n+1)=\log n+\log(n+1)-\log n$$

$$=\log n+\log(1+\frac{1}{n})=\log n+\frac{1}{n}-\frac{1}{2n^2}+\ldots.$$

Hence, $\log n=\log(n+1)-\frac{1}{n}+\frac{1}{2n^2}-\ldots+\ldots=-C+1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots$

$$+\frac{1}{n-1}+\frac{1}{n}-\frac{1}{n}+\frac{1}{2n^2}-\ldots+\ldots$$

i. e., $\log n=(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n-1}-C+\frac{1}{2n^2}-\ldots+\ldots$

We may obtain C in the form of a definite integral, as follows:

$$\phi(x)=\frac{d}{dx}\log\Gamma(1+x)\text{ (by definition) }=\frac{d}{dx}\Gamma(1+x)/\Gamma(1+x).$$

$$\therefore \phi(x)\Gamma(1+x)=\frac{d}{dx}\Gamma(1+x)=\frac{d}{dx}\int_0^\infty z^x e^{-z}dz=\int_0^\infty z^x e^{-z}\log z dz.$$

Put $x=0$; and since $\Gamma(1)=1$, we have

$$\phi(0)=\int_0^\infty e^{-z}\log z dz. \qquad \therefore C=-\phi(0)=\int_0^\infty e^{-z}\log\left(\frac{1}{z}\right)dz$$

which gives Euler's constant.

MECHANICS.

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234. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

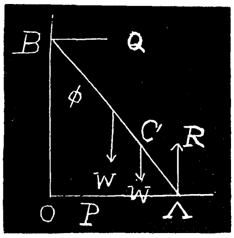
A ladder is placed with one end resting against a smooth wall and making with it an angle ϕ. Also, the roughness of the ground prevents it from slipping. A man weighing as much as the ladder ascends to the top. Taking μ as the coefficient of friction, prove:

(a) The ladder will slip before he gets to the top if $\phi>\tan^{-1}4\mu/3$.

(b) If the ascent be feasible, there will be three times as much friction when he is at the top as when he is at the bottom, (See Jeans' *Theoretical Mechanics*, p. 47.)

I. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let $AB=c$ be the ladder; let some position of the man be at C, AC being $=a$. Let the reaction at A and B be, respectively, $=P$ and Q, and the ladder be uniform, that is, its weight W acting at the mid-point of AB; the friction $P\mu$ acts along AO. Resolving along AO and BO, we have $Q=P\mu$, $P=2W$, and the equation of the moments with reference to A, $W(a+\frac{1}{2}c)\sin\phi=Qc\cos\phi=2W\mu c\cos\phi$; whence $\tan\phi=\dfrac{4c\mu}{2a+c}$. For $a=c$, $\tan\phi=\dfrac{4\mu}{3}$, $Q=\frac{1}{2}W\tan\phi$ $=\frac{2}{3}W\mu.P\mu=2W\mu$.



II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Let $2l$ be the length of the ladder, R the pressure on the ground, R' the pressure on the wall. Assuming that the man is at the top and that the ladder is about to slip, equating the sums of the vertical and horizontal forces to zero and taking moments about the foot of the ladder, we find

$$R=2W, \quad F=\mu R=R', \quad R'.2l\cos\phi=Wl\sin\phi+W.2l\sin\phi=3Wl\sin\phi.$$

Whence, $2R'=3W\tan\phi$. But $R'=2\mu W$. Therefore, $4\mu=3\tan\phi$ or $\phi=\tan^{-1}(4\mu/3)$. A larger value of ϕ makes $R'>F$, and the man cannot reach the top before slipping begins.

Friction, F, when the man is at top$=R'=\frac{3}{2}W\tan\phi$. When the man is at the bottom, the moment equation gives $R'.2l\cos\phi=Wl\sin\phi$. Whence $R'=\frac{1}{2}W\tan\phi$.

Hence, friction when at the top, is three times as great as when at the bottom.

Solved similarly by G. B. M. Zerr.

PROBLEMS FOR SOLUTION.

ALGEBRA.

331. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Extract the square root of $21+6\sqrt{2}+2\sqrt{21}-6\sqrt{3}-6\sqrt{7}-2\sqrt{6}-2\sqrt{4}$ and also of $4\sqrt{2}+2\sqrt{6}-9-4\sqrt{3}$.

332. **Proposed by C. N. SCHMALL, New York City.**

Solve the quadratic, $x^2+ax+b=0$, without completing the square.

GEOMETRY.

359. **Proposed by W. J. GREENSTREET, M. A., Stroud, England.**

Two tangents are drawn to two confocal parabolas from any point on a common tangent. Show that the former two tangents and their chord of contact envelop yet another confocal parabola.

360. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

A circular segment, area A, revolves successively about the diameters (fixed) d, d', intersecting at an angle θ. If v=volume about d, v' the volume about d', then $v^2+v'^2-2vv'\cos\theta$ is independent of the position of the segment.

361. **Proposed by W. J. GREENSTREET, M. A., Stroud, England.**

$ABCD$ is a quadrilateral. The bisectors of A and C meet in O_1; those of B and D meet in O_2. Find the tangent of the angle between AD and O_1O_2 in terms of sines and cosines of A, D, $A+B$, and $A+D$.

CALCULUS.

389. **Proposed by G. W. DROKE, Professor of Mathematics, University of Arkansas.**

Find the curve such that the rectangle under the perpendiculars from two fixed points on the normals be constant.

MECHANICS.

242. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A weight W is supported by three strings of the same size and quality lying in the same plane. The middle string is vertical, one string makes with it an angle θ on one side, and the other string makes with it an angle ϕ on the other side. Find the stresses T_1, T_2, T_3 in the strings.

243. Proposed by C. N. SCHMALL, New York City.

In a game of billiards a player observes two balls, A and B, at rest in a certain position and concludes that it would be to his advantage to project A against B in such a manner that, as a result of the impact, A might suffer the greatest possible deviation from its course. Taking the balls to be equal and smooth, each of diameter a and elasticity e, and the distance between their centers to be d, show that he can accomplish the desired motion by projecting A in a line making an angle equal to the

$$\sin^{-1}\frac{a}{d}\sqrt{\frac{1-e}{3-e}}$$

with the line joining the centers.

NOTES AND NEWS.

In connection with the series of articles on the teaching of collegiate mathematics, the paper by Professor Smith in this number is most timely and suggestive. It is greatly to be desired that we in America should deliberately take account of stock mathematically, and nothing could give so great an incentive as this opportunity to join with the other important nations in both self inspection and mutual comparison of methods and results.

In the February number will appear the third paper of the series, by Professor W. A. Granville of Yale University, on "The Teaching of the Elements of Trigonometry." S.

The twenty-sixth meeting of the Chicago Section of the American Mathematical Society was held at Chicago on December 31, 1909, and January 1, 1910. There were in attendance forty-seven members of the Society and twenty-five papers were presented. Forty members dined together on New Year's Eve and enjoyed a most interesting and profitable occasion for the promotion of acquaintance and good fellowship. The officers of the Section elected for 1910 are: Professor L. E. Dickson, chairman; Professor H. L. Slaught, secretary; and Professor W. B. Ford, third member of the program committee. Professor G. A. Miller has been chairman of the Section for the past two years, and was also secretary of Section A of the

American Association for the Advancement of Science. Professor Miller attended both the Boston and Chicago meetings and presented papers before both meetings. S.

The American Mathematical Society held its annual meeting at Boston during holiday week in affiliation with the American Association. Joint sessions were held with the Physics and Mathematics Sections and a joint dinner brought together the representatives of these sections with the New England Association of Mathematic Teachers. Eighteen papers were read before the sessions of the American Mathematical Society and seven before the joint session. S.

Professor E. H. Moore, University of Chicago, has been elected chairman of Section A of the American Association for the Advancement of Science for the meeting to be held at the University of Minnesota during the next convocation week. M.

Professor E. J. Wilczynski, University of Illinois, has recently been awarded a prize by the Belgian Royal Academy. This would appear to be the second prize in pure mathematics awarded to an American by a foreign Academy. M.

BOOKS AND PERIODICALS.

Plane Geometry Developed by the Syllabus Method. By Eugene Randolph Smith, A. M., Head of the Department of Mathematics, Polytechnic Preparatory School, Brooklyn, N. Y. (formerly Head of the Department of Mathematics, Montclair High School). 8vo. Cloth. 192 pages. New York and Chicago: American Book Co.

The object of this book seems to be to meet the demands of those teachers who wish only the skeleton of a demonstration, leaving the work of putting on the muscles and giving form and symmetry to the geometric body, to the student. Those teachers, too, who wish no text will find this book suited to their needs.

The book opens with a section on Logic. This feature of the book will be condemned by those teachers of geometry who believe that Euclidean geometry should be as much divorced from formal logic as are the other branches of mathematics.

The exercises are in two divisions, those under the theorems chosen to illustrate the use of them and those in general lists, to give the student practice in finding for himself the underlying principle.

The book will be found very suggestive for teachers of geometry. F.

Plane Trigonometry. By Edward R. Robbins, Senior Mathematical Master, The William Penn School. 8vo. Cloth. 153 pages. New York and Chicago: The American Book Co.

This contains only the most essential facts and principles of the subject. F.

A Course in Mathematics. For Students of Engineering and Applied Science. By Frederick S. Woods and Frederick H. Bailey, Professors of Mathematics in the Massachusetts Institute of Technology. Vol. II, Integral Calculus, Functions of Several Variables, Space Geometry, Differential Equations. 8vo. Cloth. xi+410 pages. Price, $2.25. Boston and Chicago. Ginn & Co.

This volume completes the plan of the course of study in mathematics outlined in the preface of the first volume. Integration of functions of a single variable is treated in the first eight chapters, emphasis being laid upon fundamental processes. In this part of the book the authors have added what seems to them to be a new feature, viz., the treatment of simple differential equations in close connection with integration long before the formal study of such equations.

With the ninth chapter begins the study of functions of two or more variables, being accompanied by the use of solid analytical geometry, and the treatment of partial differentiation and of multiple integrals. Also here is introduced a short discussion of line and surface integrals.

The latter part of the book consists of chapters on series,—Fourier, Taylor, etc.,—complex numbers and differential equations.

This, with the companion first volume, will form an excellent course in mathematics for the practical mathematician and will give the teacher of mathematics who does not wish to teach the subject in closed compartments a good outline for that kind of work. F.

Strength of Materials. An Elementary Study Prepared for the Use of Midshipmen at the U. S. Naval Academy. By H. E. Smith, Professor of Mathematics, U. S. Navy. 12mo. Cloth. ix+170 pages, 73 figures. Price, $1.25. New York: John Wiley & Sons.

The student of applied mathematics will find in this little volume a fine presentation of the subject. While the treatment is necessarily brief, yet it is clear. The problems are numerous and well selected. Teachers having only a limited time to devote to the subject, will do well to examine this book with a view to its use in their classes. F.

In Starland with a Three-Inch Telescope. By William Taylor Olcott, Author of a "Field Book of the Stars." 12mo. Cloth. xiv+146 pages. Price, $1.00. New York and London: G. P. Putnam's Sons.

This little volume will be found to be of inestimable value to teachers and students having the use of a small telescope. The constellations are grouped with the seasons for the sake of convenience and in connection with the forty groups diagramed is given a brief description of the most interesting objects to be seen in each group. F.

Elements of Plane and Spherical Trigonometry. By James Howard Gore, Ph. D., Professor of Mathematics, George Washington University, Author of Plane and Spherical Geometry, Elements of Geodesy, History of Geodesy, Bibliography of Geodesy, etc., etc. With six-place tables of logarithms of numbers from 1 to 10000 and logarithmic sines, cosines, tangents and cotangents from 0° to 90°, and an auxiliary table for small angles. 8vo. Cloth. vi+122 pages, +xvii+80 pages of tables. Price, $1.25. New York and London: G. P. Putnam's Sons.

This book does not differ essentially from many of the recent texts on the subject. F.

The Calculus and Its Applications. A Practical Treatise for Beginners, Especially Engineering Students. By Robert Gordon Blaine, M. E., Assoc. M. Inst., C. E., etc., Lecturer at the City Guild's Technical College, Finsbury, London, E. C., Author of Hydraulic Machinery, Lessons in Practical Mechanics, The Slide Rule, etc. 8vo. Cloth. ix+321 pages. Price, $1.50. New York: D. Van Nostrand & Co.

This is a good text, well written, containing numerous interesting problems, suitable for the student of pure mathematics as well as the student of applied mathematics.

The fundamental principles of the Calculus are established in a practical way and will be easily understood by the student. F.

Plane and Spherical Trigonometry. By Levi L. Conant, Ph. D., Professor of Mathematics in the Worcester Polytechaic Institute. 8vo. Cloth. 222 pages, +80 pages of tables. New York & Chicago: American Book Co.

In addition to the usual matter to be found in a text of this sort, the author has added a chapter on hyperbolic functions. The usual topics are treated with clearness and rigor. F.

Five-Place Logarithmic and Trigonometric Tables. Edited by James M. Taylor, Colgate University. 8vo. Cloth. xv+72 pages. Boston and Chicago: Ginn & Co.

A convenient feature of these tables is to be found in the patent index, greatly facilitating the work of finding the logarithms of numbers and functions of angles. F.

Applications of the Calculus to Mechanics. By E. R. Hedrick, Professor of Mathematics, University of Missouri, and O. D. Kellogg, Assistant Professor of Mathematics, University of Missouri. 8vo. Cloth. vi+116 pages. Boston and Chicago: Ginn & Co.

This book is a formulation of the work done in the Missouri State University following the course in Sophomore Calculus. It is believed that the work here outlined and pursued by the student will fix the principles of the Calculus more firmly in mind. A number of problems of some difficulty are inserted in each chapter. F.

The Fundamental Principles of Chemistry. An Introduction to all Text-Books of Chemistry. By William Ostwald. Authorized Translation by Harry W. Morse. 8vo. Cloth. xii+349 pages, with 65 figures in the text. Price, $2.25. New York: Longman's, Green & Co.

The author states in the preface that the object of this book is to present the fundamental principles of Chemistry, their meaning and connection free from irrelevant additions, and the book represents an opinion of the author that it is possible to work out a chemistry in the form of a rational scientific system without bringing in the properties of individual substances. The work is one of very great interest to the teacher of Physics as well as of Chemistry. F.

The Monist. A Quarterly Magazine devoted to the Philosophy of Science. Edited by Paul Carus. Price, $2.00 per year in advance.

The January, 1910, number contains three articles of much interest to mathematicians. The first is The Nature of Logical and Mathematical Thought, by Paul Carus; the second, The Future of Mathematics, by H. Poincare, translated by Dr. Halsted; and the third, Transfinite Numbers and the Principles of Mathematics, by Philip E. B. Jourdain. F.

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ON THE TEACHING OF THE ELEMENTS OF PLANE TRIGONOMETRY.

By W. A. GRANVILLE, Yale University.

In teaching we proceed from that which is familiar to that which is new. The successful teacher leads the student into a knowledge of Trigonometry by making the successive steps to it as gradual and natural as possible. From his knowledge of Geometry the student has usually a fair understanding of the relations between the sides and angles of a triangle; and of the different kinds of triangles, he is most familiar with the right triangle. Hence it is best to commence with the trigonometric functions of an acute angle and to define them as the ratios of the sides of a right triangle having that acute angle as one of its angles. The functions should be defined and written in pairs as reciprocals of each other in order to aid the memory and to emphasize one of the most important of their functional relations. No attempt should be made at the start to teach the student any particular definition of angle; let angles be to him just what they were to him in his Geometry. The general definition of angle should not be given until the student has mastered the right triangle. Nor is it necessary at this time to introduce circular measure (radians). Only even degrees should be given at first, then minutes or decimal parts of a degree. The use of seconds to to any considerable extent in a first course in Trigonometry has a tendency to obscure the theory, and the additional calculations involved are apt to degenerate into mere drudgery for student and teacher alike. The division of the degree into decimal parts, instead of using minutes and seconds, has much favor by expert computers. Irrespective of what the future of the Metric System may be in the United States, it seems certain that the decimal division of the degree is fast gaining ground in both theoretical and practical work. And right here I wish to record a most emphatic protest against a notation in which 36.2° is written $36^\circ.2$. There is no reason whatever for inserting the unit of measurement between the digits of a number. What would we think of the engineer who wrote 127.ft.36 instead of 127.36

ft., and how would it look for a merchant to advertise a marked down sale of straw hats from 2.$50 to 1.$49!

The success or failure of a course in any subject depends in a large measure on a few of the first lessons. This is especially true of Trigonometry. If you begin by giving the average student a general definition of the trigonometric functions, he is apt to become badly confused and easily gets discouraged. On the other hand, if the student in the first few lessons learns the solution of right triangles thoroughly (using the natural values of the functions) he already knows how to use some of the most powerful tools in Trigonometry. In fact, if the student can apply the three following rules, namely:

Side opp. an acute angle=hypot.×sine of the angle,

Side adj. an acute angle=hypot.×cosine of the angle,

Side opp. an acute angle=adj. side×tangent of the angle,

he is already able to solve a large number of the trigonometric problems occurring in the elements of pure and applied mathematics. If we think of the applications of Trigonometry in the elements of Analytic Geometry, for instance, it is surprising how little trigonometric knowledge outside of these three rules is really necessary. When the student has mastered the right triangle and has applied this knowledge to the solution of numerous practical problems he begins at once to realize the utility and beauty of Trigonometry and so at the very start becomes interested and gains a confidence that will stand him in good stead later on.

Logarithms should not be introduced until the student has mastered the fundamental principles of Trigonometry and has used the natural functions in the solution of both right and oblique triangles. If we begin by using logarithms in our calculations the student naturally associates the trigonometric functions with logarithms and after a time he finds it difficult to separate the two notions. It is very important to impress upon the student the fact that in Trigonometry logarithms are employed chiefly for the purpose of minimizing the labor connected with the computations. Beginning with the natural functions also introduces the subject of interpolation as a logical sequence.

After the right triangle has been digested the following general definition of angle should be given:

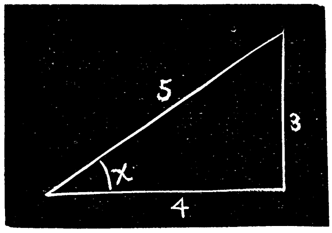
An angle may be considered as generated by a line which first coincides with one side of an angle, then revolves about the vertex, and finally coincides with the other side.

Positive and negative angles and angles of any magnitude should then be defined.

Next comes the definitions of the trigonometric functions of any angle using rectangular coordinates. The student has probably already used rectangular coordinates in his Algebra, so that these definitions will appear perfectly natural to him. If the student is not familiar with coordinates he

should be taught their use before proceeding further. Having given the value of a trigonometric function it is now easy to construct geometrically all the angles which satisfy the given value, and to find the values of the other five functions. Or, we may express any five of the functions in terms of the sixth. If the construction of the angle is not required, the process may be shortened by simply drawing a right triangle to serve as a check on the numerical part (not the algebraic signs) of the work. Thus having given $\sec x = \frac{5}{4}$, to find the value of the other five functions. Here

$$\sec x = \frac{5}{4} = \frac{\text{hypt.}}{\text{adj. leg}}$$



$$\text{opp. leg} = \sqrt{(5^2 - 4^2)} = 3.$$

The numerical values of the functions will then be:

$\sin x = \frac{3}{5}$ $\qquad$ $\csc x = \frac{5}{3}$

$\cos x = \frac{4}{5}$ $\qquad$ $\sec x = \frac{5}{4}$

$\tan x = \frac{3}{4}$ $\qquad$ $\cot x = \frac{4}{3}$

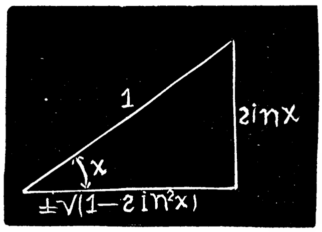
The given secant being positive, the angle will lie either in the first or the fourth quadrants. If the angle lies in the first quadrant all the functions are positive and the above results are correct as they stand. If the angle lies in the fourth quadrant, we merely change the algebraic signs as follows:

$\sin x = -\frac{3}{5}$ $\qquad$ $\csc x = -\frac{5}{3}$

$\cos x = \frac{4}{5}$ $\qquad$ $\sec x = \frac{5}{4}$.

$\tan x = -\frac{3}{4}$ $\qquad$ $\cot x = -\frac{4}{3}$

Again, express in terms of $\sin x$, the other five functions of x. Here



$$\sin x = \frac{\sin x}{1} = \frac{\text{opp. leg}}{\text{hyp.}}$$

adj. leg $= \pm\sqrt{(1 - \sin^2 x)}$. Hence

$$\sin x = \sin x \qquad \csc x = \frac{1}{\sin x}$$

$$\cos x = \pm\sqrt{(1 - \sin^2 x)} \qquad \sec x = \pm\frac{1}{\sqrt{(1 - \sin^2 x)}}$$

$$\tan x = \pm\frac{\sin x}{\sqrt{(1 - \sin^2 x)}} \qquad \cot x = \pm\frac{\sqrt{(1 - \sin^2 x)}}{\sin x}$$

The line definitions (using the unit circle) of the trigonometric functions should now be given. By means of these definitions the limits of the functions, as the angle approaches a multiple of a right triangle, can be illustrated geometrically in a manner most satisfactory to the student. The fundamental relations between the functions (except the reciprocal relations which follow at once from the ratio definitions) are obtained most naturally from the line definitions, as is also the proof of the addition theorem.

Circular measure should now be given with numerous examples illustrating its relations to degree measure. The student is now ready for the proof of

$$\underset{x=0}{\text{limit}}\frac{\sin x}{x}=1 \text{ and } \underset{x=0}{\text{limit}}\frac{\tan x}{x}=1,$$

two limits of great importance in both pure and applied mathematics. Or,

THEOREM. *We may replace $\sin x$ and $\tan x$ in our calculations by x when x is a very small angle and is expressed in circular measure.*

Attention should be called to the fact that when we wish to find the functions of angles near 0° or 90°, ordinary interpolation will in general give inaccurate results, and that the above theorem should be used instead.

In the reduction of functions of any angle to the functions of an acute angle the student should be encouraged to use the forms

$$180^\circ \pm x, \text{ or } 360^\circ \pm x,$$

for then the name of the function remains unchanged throughout the operation and there is less liability of making a mistake. And it should be pointed out that one of the principal reasons for making such a reduction is that our tables give the values of the trigonometric functions of angles from 0° to 90° only.

In a first course the student should not be required to learn the proof of the addition theorem for any but acute angles; but his attention should be called to the fact that the theorem actually does hold true for any angles, and this statement should be illustrated by examples.

Functions of twice an angle and half an angle, in terms of the functions of the angle, should now be given. More prominence than is customary should be given to the formulas

$$\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}=\frac{1-\cos x}{\sin x},$$

for they express $\tan\frac{x}{2}$ rationally in terms of $\sin x$ and $\cos x$.

In proving the four formulas usually commencing with

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$

we should emphasize the important fact that they express the algebraic sum of sines or cosines in terms of their products.

Care should be taken that the student does not get tied up to any particular set of letters, Greek or Roman, as symbols for angles, or to any particular forms of the formulas. You have probably met the boy who could prove

$$\sin X - \sin Y = 2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$

all right, but who was completely floored by

$$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = \sin\alpha - \sin\beta,$$

or who could not recognize as identical the expressions

$$2\sin^2\frac{x}{2} = 1 - \cos x, \quad \tfrac{1}{2} - \tfrac{1}{2}\cos A = (\sin\tfrac{1}{2}A)^2.$$

A certain degree of uniformity in notation and form is desirable when teaching a beginner, but when trigonometric analysis is reached in the course it is well to make a point of freely varying both notation and form.

Now comes the derivation of the general value for all the angles having the same value of a function, the introduction of inverse trigonometric functions, and the solution of trigonometric equations.

Processes should be summarized into working rules whenever practicable. As for instance, the following:

General directions for solving a trigonometric equation.

First step. If multiple angles, fractional angles, or the sums and differences of angles are involved, reduce all to functions of a single angle, and simplify.

Second step. If the resulting expressions are not readily reducible to the same function, change all the functions into sines and cosines.

Third step. Clear of fractions and radicals.

Fourth step. Change all the functions to a single function.

Fifth step. Solve for the one function now occurring in the equation, and express the general value of the angle thus found. Only such values of the angle which satisfy the given equation are solutions.

Whether or not the graphs of the functions should be given in a first course depends on the time allowed. These graphs illustrate very vividly the property of the periodicity of the trigonometric functions. After a graph has been plotted from the calculated values of a function the student should be taught how to plot the graph by purely geometric methods from the unit circle.

As soon as the law of sines, law of cosines, and law of tangents have been derived, they should be employed in the solution of some oblique triangles making use of the natural values of the functions. Areas of triangles should also be found making use of the natural values.

And now comes the theory and use of logarithms. It is astonishing how incomplete the treatment of these topics is in almost all the current text books. Unless he has the aid of a teacher the student usually runs up against a stone wall when he comes to the application of logarithms to calculations. There are many tricks in the mathematician's trade when it comes to the use of logarithms. The texts do not always explain fully these artifices of the calculator or the peculiarities of the particular tables used. With the teacher this has become second nature and he sometimes becomes impatient at the apparent stupidity of the pupil. It is as if we expected the student to be born with an instinct for the use of logarithms! The teacher should insist on having the calculations set down according to some set form or scheme. In logarithmic computations the student should always write down an outline or skeleton of the computation before using his table at all. For, it saves time to look up all the logarithms at once and, besides, it reduces the liability of error to thus separate the theoretical part of the work from that which is purely mechanical.

All results should be verified by the student himself. The importance of this has been generally overlooked by teachers. A student gains much in interest and self-confidence when he feels independent of both his book and his teacher when it comes to proving the accuracy of his results.

It is imporiant that the student should draw the figures connected with the problems as accurately as possible. This not only leads to a better understanding of the problems themselves, but also gives a clearer insight into the meaning of the trigonometric functions and makes it possible to test roughly the accuracy of the results obtained. The only instruments necessary are a graduated ruler and a protractor, and the student should be advised to use them freely.

Throughout the subject practical problems relating to matters of common knowledge should be given as far as possible. And here, at least, we should call the attention of the student to the fact that in the examples usually given it has been assumed that the given data were exact. That is, if two sides and the inclined angle of a triangle are given, as 135 ft., 217 ft., and 25.3° respectively, we have taken for granted that these numbers are not subject to errors made in measurement. This is in accordance with the

plan followed in the problems that the student has solved in Arithmetic, Algebra, and Geometry. It should not be forgotten, however, that when we apply the principles of Trigonometry to the solution of practical problems,— engineering problems, for instance,— it is usually necessary to use data which have feen found by actual measurement and therefore are subject to error. For instance, if the length of a line is measured by a steel tape, account must be taken of the expansion due to heat as well as the sagging of the tape under various tensions. And in making several measurements one should carefully see that they are made with about the same precision. Thus, it would be folly to measure one side of a triangle with much greater care than another; for, in combining these measurements in a calculation, the result would at best be no more accurate than the worst measurement. Similarly, the angles of a triangle should be measured with the same care as the sides. The number of significant figures in a measurement are supposed to indicate the care that was intended when the measurement was made. In ordinary engineering practice only the first three or four significant figures of the measurement are not subject to error. It is therefore evident that the use of five or six place tables in calculations involving these measurements introduces an unnecessary refinement and merely adds to the labor without making the results more accurate than they would be if four place tables had been used. In a large number of cases three place tables are accurate enough.

At regular intervals the student should be required to write down from memory a list of all the fundamental formulas studied.

Brief notes on the history of the science should be given by the teacher as opportunity offers.

The computation of logarithms and of the trigonometric functions from series, De Moivre's Theorem, and the hyperbolic functions, do not properly belong to a first course in Trigonometry.

FACSIMILE EDITIONS OF JOHN BOLYAI'S SCIENCE ABSOLUTE OF SPACE.

By GEORGE BRUCE HALSTED.

At last, any one has a chance to see just how looked the most extraordinary two dozen pages in the history of human thought.

How John Bolyai himself looked, the world can never know, for before it woke up to his genius he was dead of disgust and so covered with oblivion that no picture of him remains, though his love child Dyonis, son of Rosa Orbán, still lives. His eyes were blue. So much I read in his passport

at Maros-Vásárhely, where I obtained from the college in which John's father, Farkas, was professor, its copy of the priceless gem now reproduced in facsimile by Stephen Biás de Ders (Dersi Biás István).

This facsimile is the more welcome because in my translation, now in its fourth edition, are reproduced entire in Japan in English, I put Bolyai's symbols into my own words in a terminology I had already used in translating Lobachevsky, and sacrificed to uniformity the marked distinction in notation between these two independent creators of the new universe. Both recognize that through a point A outside of a straight BC there may be an infinity of straights coplanar with BC but nowhere meeting it. This was also discovered independently by Philip Kelland, Cambridge senior wrangler and tutor of the great Sylvester, whose real name was not Sylvester but plain James Joseph. Now John Bolyai and Kelland, adhering strictly to Euclid's definition of parallels, called all these intersecting straights parallel to BC. Kelland made no further distinction among them, and his work remained sterile. The others both recognized the all-importance of a boundary line, and to these boundary lines Lobachevsky restricted the application of the word parallel; so that through every point he has two intersecting parallels to the same straight, one parallel to it toward one end of it, the other parellel to it toward the other end of it, making of parallelism a *sensed* relation. For the remaining infinity of the Bolyai and Kelland parallels Lobachevsky has no word. For them I have adopted the term *ultra-parallel.* In this terminology each parallel to a straight meets it at infinity. Two straights parallel to each other have a figurative point in common. But two straights ultra-parallel to each other, though coplanar, have not even a figurative point in common. They do not meet even at infinity. Two straights parallel in opposite senses to the same straight intersect. Two straights parallel to the same straight in the same sense are parallel to one another.

Into this terminology I translated or rather paraphrased John Bolyai, and the world has accepted it. He largely wrote in symbols of his own make, which never have been used by any one else.

His symbol for that half of the straight AB which commences at the point A and contains the point B, I translate *the ray AB.*

His first section begins: "If the ray AM is not cut by the coplanar ray BN, but is cut by every ray BP within the angle ABN, this is designated by $BN \parallel\!| AM$." His symbol $\parallel\!|$ I replaced by $\parallel$ and translate *parallel.* But for Bolyai himself it meant *asymptote.* And no one word could have expressed more of the new thought. That two rays can be asymptotes to one another seemed so strange an idea we cannot sufficiently marvel at the courage, the nerve, the fineness of our young Magyar.

B. S. Teubner, of Leipzig, has on sale 300 copies of the *Ders Facsimile.* But let those who on it see the date of the original, 1832, not forget the letter I saw at Maros-Vásárhely and was the first to publish, making the date

1823 ever memorable. On its publication thus in America Charles S. Peirce wrote in *The Nation*, March 17, 1892, p. 212, in a review of Halsted's Bolyai:

> There is a winningly enthusiastic letter from Bolyai János to his father, telling him of the great step. He says: "I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you will yourself acknowledge it. At present I can not say more than that from nothing I have created a wholly new world."

Ten years later this letter was published in Hungary in Magyar and Latin. Later came the establishment of the great Bolyai prize (Prix Bolyai) by the Hungarian Academy of Sciences, a prize of ten thousand crowns founded on the occasion of the hundredth anniversary of the birth of John Bolyai "to perpetuate the memory of this illustrious scientist," whose very grave was lost, but for one woman, who told where to erect the belated granite shaft now dedicated to dauntless genius. Before the prize was first given I ventured in *Science*, as a bit of prophecy, to predict it would go to Poincaré. It did. Now it is again to be adjudged in 1910.

Attempts are being made to fix the site of Bolyai's house, No. 1004, burned in 1876. Meanwhile the quaint house of his father Farkas, redolent with memories of Hungary's wonder-child, in which I was entertained so charmingly by Professor Koncz, who died in 1906, has been torn down to make way for a new street. Franz Schmidt of Budapesth has also passed away, who on the banks of the Danube confided to me the tales his own father told him of the fiery young Captain Bolyai, whom he saw, with his treasured Damascus sword, hack off an iron spike driven into his door-post, and who, another Ivanhoe, challenged by 13 Austrian cavalry officers at once, discomfited in succession all the challengers, a swordsman so redoubtable he could have held at bay those kindred spirits D'Artagnan, Cyrano and Captain Fracasse. Cyrano composed poetry while swording his man. Bolyai played his marvellous violin between bouts while sabreing his 13 cavalry officers. But what if he had taken it into that wonderful head of his to challenge the whole Austrian army in squads of thirteen?

His violin he expressly mentions in his will. I know not the fate of Damascus blade; but now we are given in facsimile the still more trenchant instrument wherewith he severed the bonds in which throughout the ages Euclid had held captive the mind of man.

BIANGULAR COORDINATES.

By G. B. M. ZERR, Philadelphia, Pa.

It is the purpose of this discussion to set forth a very elementary exposition of a most interesting subject* rather than anything new, with the hope that it may be further developed in future issues. In what follows the notation of Professor Genese is used.†

THE STRAIGHT LINE.

We assume two fixed points A, B as poles, and determine the position of any point P in the plane of AB by the angles $PAB=\theta$ and $PBA=\phi$, regarding θ, ϕ as positive on one side of AB and negative on the other.

Let PD be a straight line intersecting AB in D, N the foot of the perpendicular from P on AB; also let $AB=c$, $AD=d$, $BD=f$, $\cot\theta=\lambda$, $\cot\phi=\mu$.

Then $AN+BN=c$... (1).

$PN=DN\tan D=(AN-d)\tan D=NB\tan\phi$... (2).

$PN=(f-NB)\tan D=AN\tan\theta$... (3).

Eliminating AN, NB from (1), (2), (3) we get

$$f\tan D\tan\phi-c\tan\theta\tan\phi-(c-f)\tan D\tan\theta=0,$$
$$\text{or } f\tan D\tan\phi-c\tan\theta\tan\phi-d\tan D\tan\theta=0.$$

Hence $f\cot\theta-d\cot\phi-c\cot D=0$. Writing p for f, q for $-d$, r for $-c\cot D$, we get for the equation to any straight line

$$\text{I.}\quad p\lambda+q\mu+r=0.$$

Hence $\lambda=0$, $\mu=0$ represents straight lines perpendicular to AB, through A and B, respectively. Let $p_1\lambda+q_1\mu+r_1=0$ be any other line, α the angle line I makes with this last line. Then, since $r=-c\cot D$, $r_1=-c\cot D_1$,

$$\cot\alpha=\frac{(r/c)(r_1/c)+1}{r/c-r_1/c}=\frac{rr_1+c^2}{c(r-r_1)}.$$

Therefore $\cot\alpha=0$ or ∞ according as the lines are perpendicular or parallel. If $r_1=-c^2/r$, the lines are perpendicular. If $r_1=r$, the lines are parallel.

*A consideration of biangular coordinates may be found in various sources, including the following: *Quarterly Journal of Mathematics*, Volumes IX and XIII; Carr's *Synopsis of Pure Mathematics;* Milne's *Companion to Weekly Problem Papers.*

†See *Weekly Problem Papers.*

Let AB be the axis of abscissas, and the perpendicular bisector of AB, the axis of ordinates, then, it follows at once that $\lambda=(c+2x)/2y$, $\mu=(c-2x)/2y$.

Hence, $x=\frac{c}{2}(\lambda-\mu)/(\gamma+\mu)$, $y=c/(\lambda+\mu)$... (4).

The values of x and y from (4) in the general Cartesian equation gives

$$(Ac+2C)\lambda+(2C-Ac)\mu+2Bc=0.$$

This is of the form

$$p\lambda+q\mu+r=0, \text{ the same as I.}$$

(4) is used to transform from rectangular to biangular coordinates.

Let $\left.\begin{array}{l} p\lambda+q\mu+r=0 \\ p\lambda_1+q\mu_1+r=0 \\ p\lambda_2+q\mu_2+r=0 \end{array}\right\}$ represent the same line.

Eliminating p, q, r, we get

$$\text{II.} \quad \lambda-\lambda_2=\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}(\mu-\mu_2)$$

for the equation to the line joining (λ_1, μ_1), (λ_2, μ_2).

As a simple illustration let it be required to find the locus of the vertex of a triangle when the base and the difference of the squares of the other sides are given.

Here $AB=c$, $AP^2-BP^2=m^2$. But $AP^2-BP^2=AN^2-BN^2=c(AN-BN)=m^2$. Also $\lambda=AN/PN$, $\mu=BN/PN$.

Hence, $c(\lambda-\mu)/PN=m^2$, $PN=c/(\lambda+\mu)$.

Therefore, $(c^2-m^2)\lambda=(c^2+m^2)\mu$, a straight line perpendicular to AB, and dividing it in the ratio $(c^2+m^2):(c^2-m^2)$.

The Conic Sections.

(a) By (4) the rectangular equation $\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1$ transforms into the biangular equation

$$b^2(4a^2-c^2)(\lambda^2+\mu^2)+2b^2(4a^2+c^2)\lambda\mu\mp 4a^2c^2=0\text{... (5).}$$

When $c=2a$, (5) becomes $\lambda\mu=\pm a^2/b^2$... (6), an ellipse or hyperbola, according as we use the plus sign or the minus sign, with AB as major axis.

When $c=2ae$, (5) becomes

$$b^2(1-e^2)(\lambda^2+\mu^2)+2b^2(1+e^2)\lambda\mu\mp 4a^2e^2=0,$$

$$\text{or } \lambda^2+\mu^2+\frac{2(1+e^2)}{1-e^2}\lambda\mu = \pm\frac{4a^2e^2}{b^2(1-e^2)}\ldots(7),$$

the equation for the ellipse and the hyperbola, when A, B are the foci.

If $a=b$, (6) becomes $\lambda\mu=1$ for the circle and $\lambda\mu=-1$ for the rectangular hyperbola, and (7) becomes $\lambda+\mu=0$, the line at infinity. In this last case A and B coincide and hence $c=0$. (6) is found geometrically as follows:

The geometry of the ellipse and the hyperbola gives us $PN^2:AN.NB=b^2:a^2$. As before stated, $\mu=NB/PN$, $\lambda=\pm AN/PN$.

Hence $\lambda\mu=\pm AN.NB/PN^2=\pm a^2/b^2$. From II, the secant through two points on (6) is $\lambda-\lambda_2=\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}(\mu-\mu_2)$.

But $\lambda_1\mu_1=\lambda_2\mu_2$ or $\lambda_1/\lambda_2=\mu_2/\mu_1$. Hence, $(\lambda_1-\lambda_2)/\lambda_2=-(\mu-\mu_2)/\mu_1$, or $\lambda-\lambda_2=-(\lambda_2/\mu_1)(\mu-\mu_2)$.

For the tangent at (λ_1, μ_1) we have $\lambda_1=\lambda_2$, $\mu_1=\mu_2$, and hence $\lambda-\lambda_1=-(\lambda_1/\mu_1)(\mu-\mu_1)$, or $\lambda/\lambda_1+\mu/\mu_1=2\ldots(8)$.

For the circle $\lambda_1\mu_1=1$ and the tangent is given by $\lambda\mu_1+\mu\lambda_1=2$.

(b) Let c be the distance from the focus B to the directrix A of the ellipse or the hyperbola, then if the mid-point of AB is the origin, the rectangular equation is

$$y^2+(x-\tfrac{1}{2}c)^2=e^2(x+\tfrac{1}{2}c)^2$$

which reduces to

$$4y^2+4x^2(1-e^2)-4cx(1+e^2)+c^2(1-e^2)=0\ldots(9).$$

The substitution of (4) in (9) gives $e^2\lambda^2=\mu^2+1\ldots(10)$.

This result is found geometrically as follows: P is a point on the curve, R a point on the directrix, PR is parallel to BA.

Then $BP=ePR=ePA\cos\theta$. Then $BP/PA=\sin\theta/\sin\phi=e\cos\theta$, or $\operatorname{cosec}\phi=e\cot\theta$. Hence, $1+\mu^2=e^2\lambda^2$.

For two points on this curve we easily get $e^2\lambda_1^2-\mu_1^2=e^2\lambda_2^2-\mu_2^2=1$, or $\frac{\lambda_1-\lambda_2}{\mu_1-\mu_2}=\frac{\mu_1+\mu_2}{e^2(\lambda_1+\lambda_2)}$. Hence, $\lambda-\lambda_2=\frac{\mu_1+\mu_2}{e^2(\lambda_1+\lambda_2)}(\mu-\mu_2)\ldots(11)$, is the secant through (λ_1, μ_1), (λ_2, μ_2). For the tangent $\lambda_1=\lambda_2$, $\mu_1=\mu_2$.

Hence, $\lambda-\lambda_1=\frac{\mu_1}{e^2\lambda_1}(\mu-\mu_1)$ is the tangent at (λ_1, μ_1). This further reduces to $e^2\lambda\lambda_1-\mu\mu_1=e^2\lambda_1^2-\mu_1^2=1$. That is, $e^2\lambda\lambda_1-\mu\mu_1=1$ is the tangent.

(c) If AB is a chord of a circle, P the angle inscribed opposite AB, then $\theta+\phi=\pi-P$, $\cot(\theta+\phi)=-\cot P$.

Hence $\lambda\mu+(\lambda+\mu)\cot P=1$ is the equation to the circum-circle of APB.

If $P=\frac{1}{2}\pi$, $\cot P=0$, and $\lambda\mu=1$, as before stated.

(d) Let $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$ be the rectangular equation for the general conic. Substituting the values of x and y from (4) this reduces to the form

$$\text{III.}\quad a\lambda^2+2h\lambda\mu+b\mu^2+2g\lambda+2f\mu+e=0.$$

The tangent to this conic at the point (λ_1, μ_1) is

$$\text{IV.}\quad a\lambda_1\lambda+h(\lambda_1\mu+\mu_1\lambda)+b\mu_1\mu+g(\lambda+\lambda_1)+f(\mu+\mu_1)+e=0.$$

If $a=b=0=g=f$, III becomes $\lambda\mu+$a constant$=0$; IV becomes $\lambda_1\mu+\mu_2\lambda$ $+$a constant$=0$.

If $a=b=0$, we get the conic referred to any chord. Its equation is of the form

$$\text{V.}\quad \lambda\mu+g_1\lambda+f_1\mu+e_1=0.$$

Its tangent at (λ_1, μ_1) is of the form

$$\text{VI.}\quad \lambda_1\mu+\mu_1\lambda+g_1(\lambda+\lambda_1)+f_1(\mu+\mu_1)+e_1=0.$$

V and VI can be written as follows:

$$\begin{aligned}
&(\lambda+f_1)(\mu+g_1)=\text{a constant},\\
&(\lambda+f_1)(\mu_1+g_1)+(\lambda_1+f_1)(\mu+g_1)=\text{a constant}\\
&\qquad=2(\lambda_1+f_1)(\mu_1+g_1).
\end{aligned}$$

Hence, $\dfrac{\lambda+f_1}{\lambda_1+f_1}+\dfrac{\mu+g_1}{\mu_1+g_1}=2$ is a neater form for VI.

An Application.

(e) As an application of biangular coordinates, suppose we have given the base AB of a triangle and the locus of the vertex, to find the locus of the symmedian point K.

Let the vertex C describe the straight line, $p\cot A+q\cot B+r=0$... (13).

Let $\cot A=P$, $\cot B=Q$. Since the median AM and the symmedian AK are equally inclined to the bisector of A,

$$\angle MAB+\angle KAB=\angle A,\text{ or }\cot KAB=\lambda=\frac{\cot A\cot MAB+1}{\cot MAB-\cot A},$$

$$\text{and }\lambda=\frac{2P^2+PQ+1}{P+Q},\quad \mu=\frac{2Q^2+PQ+1}{P+Q}.$$

$$(P+Q)(\lambda-\mu)=2(P^2-Q^2),\text{ or }P=\frac{\lambda-\mu}{2}+Q,\quad Q=P-\frac{\lambda-\mu}{2}.$$

The value of Q substituted in the value of λ gives

$$6P^2-(5\lambda-\mu)P=\mu\lambda-\lambda^2-2, \text{ or } P=\frac{5\lambda-\mu\pm\sqrt{(\lambda^2+14\lambda\mu+\mu^2-48)}}{12}.$$

$$\text{Similarly, } Q=\frac{5\mu-\lambda\pm\sqrt{(\lambda^2+14\lambda\mu+\mu^2-48)}}{12}.$$

These values of P and Q substituted in (13) gives for the required locus

$$[(5p-q)\lambda+(5q-p)\mu+12r]^2=(p+q)^2)(\lambda^2+14\lambda\mu+\mu^2-48),$$
$$\text{or } (2p^2-pq)\lambda^2-2(p^2+q^2-pq)\lambda\mu+(2q^2-pq)\mu^2$$
$$+2r(5p-q)\lambda+2r(5q-p)\mu+12r^2+4(p+q)^2=0.$$

That is, $a\lambda^2+2\lambda\mu h+b\mu^2+2g\lambda+2f\mu+e=0$, which is the equation to the general conic.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

327. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

The coefficients of the algebraical equation $f(x)=0$ are all integers. Show that if $f(0)$ and $f(1)$ are both odd numbers, the equation can have no integral roots.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let the equation be $a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n=0$.

Then by hypothesis, a_n, and also $(a_0+a_1+a_2+\ldots+a_{n-1}+a_n)$ are odd numbers. By subtraction, $(a_0+a_1+\ldots+a_{n-1})$ is an even number; hence, either all these coefficients are even numbers, or there is an even number of odd coefficients.

Suppose (I) that $a_0, \ldots, a_{n-1}$ are all even. Then the substitution for x of *any* integer will give an even result for these terms, which cannot combine with the odd a_n to vanish.

Suppose (II) that an even number of coefficients $a_0, \ldots, a_{n-1}$ are odd integers. The substitution of any even integer for x cannot satisfy the equation, just as in (I). Any power of an odd integer (substituted for x) is odd, so that for each odd coefficient, one odd term is obtained. But there is an even number of these, besides a_n. Thus it is again impossible to have the equation satisfied by an integer.

328. **Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.**

If $x^2+xy+y^2=3a^2$, find the maximum value of $bx+cy$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

$x^2+xy+y^2=3a^2$, $bx+cy=$maximum.

$$\therefore \frac{dx}{dy}=-\frac{x+2y}{2x+y}=-\frac{c}{b}, \text{ or } y=\frac{(2c-b)x}{2b-c}.$$

$$\therefore x^2=\frac{a^2(2b-c)^2}{b^2-bc+c^2}; \text{ and } x=\pm\frac{a(2b-c)}{\sqrt{(b^2-bc+c^2)}},\ y=\pm\frac{a(2c-b)}{\sqrt{(b^2-bc+c^2)}}.$$

$$\therefore bx+cy=2a\sqrt{(b^2-bc+c^2)}=2a\sqrt{\frac{b^3+c^3}{b+c}} \text{ is a maximum.}$$

Also solved by F. L. Griffin and S. G. Barton.

329. **Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.**

Between the quantities a and b there are inserted n arithmetical and n harmonical means, and a series of n terms is formed by dividing each arithmetical by the corresponding harmonical mean. Show that the sum of the series is, $n\left[1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right]$.

Solution by HOWARD C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Neb., and S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

The n arithmetic means between a and b are:

$$(1)\ \frac{b+na}{n+1},\ \frac{2b+(n-1)a}{n+1},\ \dots,\ \frac{rb+(n-r+1)a}{n+1},\ \dots$$

The n harmonic means between a and b are:

$$(2)\ \frac{ab(n+1)}{nb+a},\ \frac{ab(n+1)}{(n-1)b+2a},\ \dots,\ \frac{ab(n+1)}{(n-r+1)b+(r-1)a},\ \dots$$

Dividing the terms of (1) by the corresponding terms of (2),

$$\frac{(b+na)(a+nb)}{ab(n+1)^2},\ \frac{[2b+(n-1)a][(n-1)b+2a]}{ab(n+1)^2},\ \dots,$$

$$\frac{[rb+(n-r+1)a][(n-r+1)b+2a]}{ab(n+1)^2}, \ldots, \text{ the series to be summed.}$$

Hence $S=\dfrac{ab(n^2+1)+n(a^2+b^2)}{ab(n+1)^2}+\dfrac{ab[(n-1)^2+r^2]+2(n-1)(a^2+b^2)}{ab(n+1)^2}+\ldots$

$$+\frac{ab[(n-r+1)^2+r^2]+r(n-r+1)(a^2+b^2)}{ab(n+1)^2}+\ldots$$

$$=\frac{1}{ab\,(n+1)^2}\{ab[n^2+(n-1)^2+\ldots+1^2]+ab[1^2+2^2+\ldots+n^2]$$

$$(a^2+b^2)[n+2(n-1)+3(n-2)+\ldots+r(n-r+1)+\ldots+n]\}=$$

$$=\frac{1}{ab\,(n+1)^2}\left[2ab\,\frac{n(n+1)(2n+1)}{6}+(a^2+b^2)\left(\frac{n(n^2+1)}{2}-\frac{n(n^2-1)}{3}\right)\right]$$

$$=\frac{1}{ab\,(n+1)^2}\left[2ab\,\frac{n(n+1)(2n+1)}{6}+(a^2+b^2)\left(\frac{n(n+1)\;(n+2)}{6}\right)\right]$$

$$=\frac{n}{ab\,(n+1)^2}\left[\frac{6ab\;(n+1)^2}{6}-\frac{n^2+3n+2}{6}\cdot 2ab+\frac{(n+1)(n+2)}{6}(a^2+b^2)\right]$$

$$=n\left[1+\frac{a^2-2ab+b^2}{6ab\,(n+1)^2}\cdot(n+1)\;(n+2)\right]=n\left[1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right].$$

Solved similarly by G. B. M. Zerr, and J. Scheffer.

GEOMETRY.

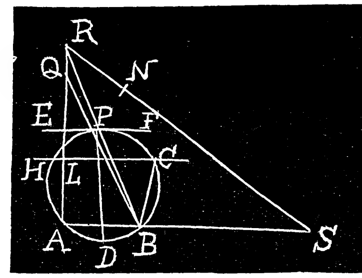
352. Proposed by G. I. HOPKINS, Professor of Astronomy, High School, Manchester, N. H.

Required, to construct the triangle, having given the base, vertical angle and sum of the altitude and the two remaining sides.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $AB=a$ be the given base; ACB the given vertical angle; p=to the sum of the altitude and the remaining sides. On AB describe the seg-

ment APB containing the given angle. Draw the diameter PD perpendicular to AB. Also draw AQ perpendicular to AB. Draw BP meeting AQ in Q. With B as a center, and a radius equal to p, describe an arc cutting AQ produced in R. With R as a center, and a radius equal to $p+AQ$, describe an arc cutting AB produced in S. On SR measure off SN=to SA. Then RN =the altitude. Take $AL=RN$, and draw



HLC parallel to AB, cutting the circle in C. Draw AC, BC. Then ACB is the required triangle. For let x, y, z be the sides BC, AC, and the altitude. Then $xy\sin C=az$, $x+y+z=p$, $a^2=x^2+y^2-2xy\cos C$.

$\therefore a^2+2xy(1+\cos A)=(p-z)^2$. $\therefore z^2-2(p+a\cot\frac{1}{2}C)=a^2-p^2$.

$\therefore z=p+a\cot\frac{1}{2}C-\sqrt{[(p+a\cot\frac{1}{2}C)^2-(p^2-a^2)]}$.

$\angle AQB=\frac{1}{2}C$, $AQ=a\cot\frac{1}{2}C$, $RS=p+a\cot\frac{1}{2}C$, $AR=\sqrt{(p^2-a^2)}$, $AS=\sqrt{[(p+a\cot\frac{1}{2}C)^2-(p^2-a^2)]}$.

$\therefore RN=z$. $\therefore$ The triangle ACB contains all the required parts.

Also solved by J. Scheffer.

353. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

In a given circle place two chords which shall be in a given ratio and also a given distance apart.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let the ratio $m:n$ to distance apart$=d$; the radius of the given circle $=r$; u, v the distances of the chords from the center.

Then $2\sqrt{(r^2-u^2)}$, $2\sqrt{(r^2-v^2)}$ are the lengths of the chords.

$\therefore m\sqrt{(r^2-v^2)}=n\sqrt{(r^2-u^2)}$, or $(m^2-n^2)r^2=m^2v^2-n^2u^2$, and $u+v=d$.

$$\therefore u=\frac{m^2d-\sqrt{[r^2(m^2-n^2)^2+m^2n^2d^2]}}{m^2-n^2},$$

$$v=\frac{\sqrt{[r^2(m^2-n^2)^2+m^2n^2d^2]}-n^2d}{m^2-n^2}.$$

Hence, if $AB=d$, take $AC=u$, $CB=v$, and with C as center and radius r describe a circle, through A and B perpendicular to AB draw lines intersecting this circle. The chords of the circle formed by these lines are the chords required.

Also solved by S. A. Corey.

354. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Find the condition that triangles which are circumscribed to one of two confocal parabolas may be inscribed in the other.

Solution by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.

Let $y^2=4a(x+a)$, $(y\cos\beta+x\sin\beta)^2=4A(x\cos\beta-y\sin\beta+A)$ be the confocal parabolas.

(1) $y^2-4ax-4a^2=0$.
(2) $y^2\cos^2\beta+x^2\sin^2\beta+2xy\sin\beta\cos\beta-4Ax\cos\beta+4Ay\sin\beta-4A^2=0$.

Calculating the invariants for (1) and (2) we get

$$\triangle=-4a^2,\quad \Theta=-4a(a+2A\cos\beta),$$
$$\Theta'=-4A(A+2a\cos\beta),\quad \triangle'=-4A^2.$$

The condition is given by $\Theta^2=4\triangle\Theta'$.

$\therefore 16a^2(a+2A\cos\beta)^2=16a^2A(A+2a\cos\beta)$.

$\therefore a^2/A^2+2(a/A)\cos\beta+4\cos^2\beta=1$, $a/A=-\cos\beta\pm\sqrt{(1-3\cos^2\beta)}$.

$\therefore \cos\beta>1/\sqrt{3}$.

$\therefore \beta$ lies between 54° 44′ and 125° 16′, and also between 234° 44′ and 305° 16′.

If $\beta=\frac{1}{3}\pi$, $a=\infty A$ or $-A$.
If $\beta=\frac{1}{2}\pi$, $a=A$ or $-A$.
If $\beta=\frac{2}{3}\pi$, $a=A$ or ∞A, etc.

CALCULUS.

282. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

A rectangular beam of length l and width w is taken horizontally from a hall of width b into a corridor at right angles to the hall. Find the width of the smallest corridor into which it can be taken.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and J. M. ARNOLD, Crompton, R. I.

Let $ABCD$ be the beam, the corner A against the hall wall, the corner B against the corrider wall; the point P of the beam against the corner of of meeting of hall and corrider.

Let $AB=l$, $BC=w$, $x=$width of corrider, PQ the portion of the width the corrider under the beam, QR the remainder of the width.

Then $PQ+QR=x$, $AQ+QB=l$, $ER=b$, $\angle BAE=\theta$. Then $PQ=w\operatorname{cosec}\theta$, $AQ=b\operatorname{cosec}\theta$, $QR=(l-b\operatorname{cosec}\theta)\cos\theta$.

$\therefore x=w\operatorname{cosec}\theta+(l-b\operatorname{cosec}\theta)\cos\theta \ldots (1)$.

Differentiating (1) we get, $w\operatorname{cosec}\theta\cot\theta+l\sin\theta=b\operatorname{cosec}^2\theta$.

$\therefore w\cos\theta=b-l\sin^3\theta\ldots(2)$.

The value of θ from (2) in (1) gives the width x required.

(2) becomes $l^2\sin^6\theta-2bl\sin^3\theta+w^2\sin^2\theta+b^2-w^2=0$.

Also solved by J. E. Sanders and the Proposer.

283. Proposed by B. F. FINKEL, Ph. D., Drury College.

By means of the calculus, determine the angle of minimum deviation of a ray of monochromatic light in passing through a triangular prism.

Solution by C. N. SCHMALL, New York City.

We have, with the usual notation,

$$\sin\phi=\mu\sin\phi' \ldots (1),$$
$$\sin\psi=\mu\sin\psi' \ldots (2),$$
$$\phi'+\psi'=\text{a constant, } k, \text{ say} \ldots (3),$$
$$u=\phi+\psi=\text{a minimum} \ldots (4).$$

Solving (1) and (2), we have, $\phi'=\sin^{-1}(\sin\phi/\mu)$, $\psi'=\sin^{-1}(\sin\psi/\mu)$. Substituting in (3), we have

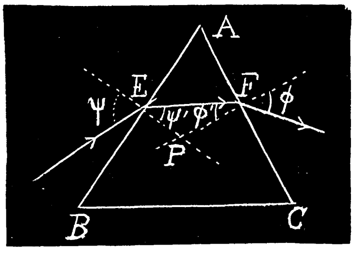
$$\sin^{-1}\left(\frac{\sin\phi}{\mu}\right)+\sin^{-1}\left(\frac{\sin\psi}{\mu}\right)=k \ldots (5).$$

Differentiating with respect to ϕ and reducing slightly, we have

$$\frac{\cos\phi}{\sqrt{(\mu^2-\sin^2\phi)}}+\frac{\cos\psi}{\sqrt{(\mu^2-\sin^2\psi)}}\frac{d\phi}{d\psi}=0 \ldots (6).$$

Differentiating (4), we have

$$\frac{du}{d\phi}=1+\frac{d\psi}{d\phi}=0 \ldots (7).$$



Equating the values of $\frac{d\psi}{d\phi}$, found from (6) and (7), we have finally,

$$\frac{\cos^2\phi}{\mu^2-\sin^2\phi}=\frac{\cos^2\psi}{\mu^2-\sin^2\psi}. \quad \therefore \phi=\psi.$$

Now, the deviation is $D=(\psi-\psi')+(\phi-\phi')=2(\phi-\phi')$, (since $\psi=\phi$)...(8).

But this will not be the case unless the $\triangle ABC$ is isosceles. Then, if θ be the angle of the prism, that is, angle A, we have

$$\phi'=\tfrac{1}{2}\theta, \text{ and from (1), } \mu=\frac{\sin\phi}{\sin\phi'}\ldots(9),$$

but from (8), $\phi=\frac{1}{2}(D+2\phi')=\frac{1}{2}(D+\theta)$. Hence, substituting in (9), we have,

$$\mu=\frac{\sin\frac{1}{2}(D+\theta)}{\sin\frac{1}{2}\theta}\ldots(10),$$

which gives the relation between the minimum deviation D, the angle of the prism θ, and the index of refraction μ.

Also solved by G. B. M. Zerr and J. Scheffer.

284. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

Inscribe the triangle of maximum area in a given circle.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let the diameter, of length d, through any vertex make angles α and β with the two sides of the triangle meeting there. Then two sides of the triangle are $d\cos\alpha$ and $d\cos\beta$; and their included angle is $\alpha+\beta$. [$\alpha-\beta$ is excluded, since a triangle contained in a semi-circle clearly can not be the maximum.] Now the area is $\frac{1}{2}(d\cos\alpha).(d\cos\beta).\sin(\alpha+\beta)$; so that we have to render a maximum the function $F(\alpha, \beta)\equiv\cos\alpha.\cos\beta.\sin(\alpha+\beta)$.

$$\text{But } \frac{\partial F}{\partial\alpha}=\cos\beta\,(\cos\alpha.\cos\overline{\alpha+\beta}-\sin\alpha.\sin\overline{\alpha+\beta})=\cos\beta.\cos(2\alpha+\beta),$$

$$\frac{\partial F}{\partial\beta}=\cos\alpha.\cos(2\beta+\alpha).$$

Equating these to zero, we obtain since $\cos\alpha\neq0$, $\cos\beta\neq0$ if there is actually a triangle:

$$2\alpha+\beta=\tfrac{1}{2}\pi,\ 2\beta+\alpha=\tfrac{1}{2}\pi, \text{ whence } \alpha=\beta=\tfrac{1}{6}\pi.$$

$$\text{Also, } \frac{\partial^2F}{\partial\alpha^2}=-2\sin(2\alpha+\beta)\cos\beta,\ \frac{\partial^2F}{\partial\alpha\,\partial\beta}=-\sin\overline{2\alpha+2\beta},$$

$$\frac{\partial^2F}{\partial\beta^2}=-2\sin(2\beta+\alpha).\cos\alpha; \text{ whence, } \left[\frac{\partial^2F}{\partial\alpha^2}.\frac{\partial^2F}{\partial\beta^2}-\left(\frac{\partial^2F}{\partial\alpha\,\partial\beta}\right)^2\right]_{\alpha=\beta=\frac{1}{6}\pi}=\tfrac{3}{2},$$

the positive result together with the negative value of $\partial^2F/\partial\alpha^2$ showing a maximum value of $F(\alpha, \beta)$.

For $\alpha=\beta=\frac{1}{6}\pi$, the sides $d\cos\alpha$, $d\cos\beta$ are equal, and the isosceles triangle is equilateral, having its vertex angle equal to $\frac{1}{3}\pi$.

Also solved by H. C. Feemster, Wilmer Thompson, G. B. M. Zerr, and C. N. Schmall.

285. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

If R_1 and R_2 are the radii of curvature of an ellipse at the extremities of a pair of conjugate diameters, show that $R_1^{2/3}+R_2^{2/3}=\dfrac{a^2+b^2}{(ab)^{2/3}}$, where a, b, are the semi-axes.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $x^2/a^2+y^2/b^2=1$ be the ellipse. Also let $x=a\cos\theta$, $y=b\sin\theta$.

$$dx=-a\sin\theta\, d\theta,\quad d^2x=-a\cos\theta\, d\theta^2-a\sin\theta\, d^2\theta,$$
$$dy=b\cos\theta\, d\theta,\quad d^2y=-b\sin\theta\, d\theta^2+b\cos\theta\, d^2\theta.$$

$$\therefore R_1=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y-dy\,d^2x}=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}{ab}.$$

$$R_2=\frac{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}}{ab}.\quad \text{But } \phi=\theta+\tfrac{1}{2}\pi.$$

$$\therefore R_2=\frac{(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{3}{2}}}{ab}.$$

$$R_1^{2/3}+R_2^{2/3}=\frac{(a^2+b^2)\,(\sin^2\theta+\cos^2\theta)}{(ab)^{2/3}}=\frac{a^2+b^2}{(ab)^{2/3}}.$$

II. Solution by HOWARD C. FEEMSTER, Professor of Mathematics, York College, York, Neb.; S. G. BARTON, Professor of Mathematics, Clarkson School of Technology, Potsdam, N. Y., and J. SCHEFFER, Hagerstown, Md.

Let x_1, y_1 be the point of intersection of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, with a diameter, and $-\dfrac{ay_1}{b}$, $\dfrac{bx_1}{a}$, the point of intersection of the ellipse with the diameter conjugate to the first.

But the radius of curvature of any point on a curve equals:

$$(1)\quad \rho=\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

For the ellipse,

$$(2)\ \frac{dy}{dx}=-\frac{b^2x}{a^2y},\ \text{and}\ (3)\ \frac{d^2y}{dx^2}=-\frac{b^4}{a^2\,y^3}.$$

From (1), (2), and (3), making the sign positive,

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

Hence, for $(x_1,\ y_1)$, $\rho=R_1=\dfrac{(a^4y_1^2+b^4x_1^2)^{\frac{3}{2}}}{a^4b^4}$.

And for $-\dfrac{ay_1}{b},\ \dfrac{bx_1}{a},\ \rho=R_2=\dfrac{(a^2b^2x_1^2+a^2b^2y_1^2)^{\frac{3}{2}}}{a^4b^4}$.

$$\therefore\ R_1^{\frac{2}{3}}+R_2^{\frac{2}{3}}=\frac{(a^2+b^2)\ (b^2x_1^2+a^2y_1^2)}{(ab)^{\frac{8}{3}}}=\frac{a^2+b^2}{(ab)^{\frac{2}{3}}}.$$

MECHANICS.

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235. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

A uniform heavy rod turns freely round a hinge at one end and rests with the other against a rough vertical wall, at angle, α, to the wall. Find the angle of arc on which this end may rest, and the pressures at the ends of the arc.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $2a$=length of rod; w, its weight; O, the hinge; OBC, the plane of the rod (vertical) perpendicular to the wall; α=angle BOC; μ=coefficient of friction between rod and wall; OA, the position of the rod at any time; OD, the projection of OA on the plane OBC; θ=angle DOA; ϕ=angle DOC; R= reaction or pressure at wall; OA, the axis of x; z, normal to AOD; y perpendicular to both x and z.

Then $OD=2a\cos\theta$, $OC=2a\sin\alpha$, $\phi=\cos^{-1}(\sin\alpha/\sin\theta)$.

The weight of the rod acting at its mid-point is equivalent to $w\cos\phi$ parallel to z, and $w\sin\phi$ parallel to OD. R is perpendicular to both OA and AD. Taking moments about y and z, respectively, we have, $aw\cos\phi=2aR$, $aw\sin\phi\sin\theta=2a\mu R$.

$\therefore \sin\theta=\mu\cot\theta=\mu\sin\alpha/\surd(\cos^2\theta-\sin^2\alpha)$.

$\therefore \sin^2\theta=\frac{1}{2}[\cos^2\alpha\pm\surd(\cos^4\alpha-4\mu^2\sin^2\alpha)]=P$. Use positive sign.

$R=(w/2)\cos\phi=[w\sin\alpha]/[2\surd(1-P)]$.

236. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at the ends, is loaded with c pounds per running foot at the ends and increases uniformly to the center, where it is b pounds per running foot. Find deflection at center due to this load.

Solution by J. E. SANDERS, Weather Bureau, Chicago, Ill., and the PROPOSER.

The load is represented by a double trapezoid, bases b, c; altitude a for each. Let $AB=2a$, $AF=FB=a$, $AE=BC=c$, $DF=b$, G the place of any point distant $AG=x$ from the left end. Draw GH parallel to AE, EK parallel to AG. Then $EK=AG=x$, $GK=AE=c$, $HG=c+[(b-c)x]/a$.

Area $AGHE=x(2ac+bx-cx)/2a$, area $ABCDE=a(b+c)$, area $AGKE=cx$, area $EKH=(bx^2-cx^2)/2a$.

Let $z=$distance of center of gravity of $AGHE$ from GH.

Then $z(\text{area } AGHE)=\frac{1}{2}x(\text{area } AGKE)+\frac{1}{3}x(\text{area } EKH)$.

$\therefore zx(2ac+bx-cx)/2a=\frac{1}{2}cx^2+(bx^3-cx^3)/6a$.

$$\therefore z=\frac{3acx+bx^2-cx^2}{3(2ac+bx-cx)}.$$

Bending moment$=\frac{1}{2}a(b+c)x-z(\text{area } AGHE)$.

$\therefore EId^2y/dx^2=\frac{1}{2}a(b+c)x-\frac{1}{6}(3acx^2+bx^3-cx^3)/a$.

$$EIdy/dx=\tfrac{1}{4}a(b+c)x^2-\tfrac{1}{6}cx^3-\frac{bx^4}{24a}+\frac{cx^4}{24a}+C.$$

When $x=a$, $dy/dx=0$. $\therefore C=-\frac{5}{24}a^3b-\frac{1}{8}a^3c$.

$$\therefore EIdy/dx=\tfrac{1}{4}abx^2+\tfrac{1}{4}acx^2-\tfrac{1}{6}cx^3-\frac{bx^4}{24a}+\frac{cx^4}{24a}-\tfrac{5}{24}a^3b-\tfrac{1}{8}a^3c.$$

$$\therefore EIy=\tfrac{1}{12}abx^2+\tfrac{1}{12}acx^2-\tfrac{1}{24}cx^4-\frac{bx^5}{120a}+\frac{cx^5}{120a}-\tfrac{5}{24}a^3bx-\tfrac{1}{8}a^3cx.$$

No constant being added as x and y are zero together.

When $x=a$, $EIy=-\frac{2}{15}a^4b-\frac{3}{40}a^4c$. If $c=0$, $EIy=-\frac{2}{15}a^4b$. If $c=b$, $EIy=-\frac{5}{24}a^4b$.

If the load consisted of b pounds per running foot at the ends and decreased to c pounds per running foot at the middle, then z times area $AGHE=\frac{1}{2}x^2.HG+\frac{2}{3}x(\text{area } EKH)$ where $AE=b$ and K is on AE instead of GH as before; $HG=(ab-bx+cx)/a$; area $AGHK=(abx-bx^2+cx^2)/a$.

$z=\dfrac{3abx-bx^2+cx^2}{3(2ab-bx+cx)}$, the same as above by replacing c with b.

$\therefore$ Bending moment$=\frac{1}{2}a(b+c)x-(3abx^2-bx^3+cx^3)/6a$.

$\therefore EId^2y/dx^2=\frac{1}{2}a(b+c)x-(3abx^2-bx^3+cx^3)/6a$.

$$EIdy/dx=\tfrac{1}{4}a(b+c)x^2-\tfrac{1}{6}(bx^3-\frac{bx^4}{4a}+\frac{cx^4}{4a})+C.$$

When $x=a$, $dy/dx=0$; $\therefore C=-\frac{1}{8}a^3b-\frac{5}{24}a^3c$.

$$\therefore EIdy/dx=\tfrac{1}{4}a(b+c)x^2-\tfrac{1}{6}bx^3+\frac{bx^4}{24a}-\frac{cx^4}{24a}-\tfrac{1}{8}a^3b-\tfrac{5}{24}a^3c.$$

$$EIy=\tfrac{1}{12}abx^3+\tfrac{1}{12}acx^3-\tfrac{1}{24}bx^4+\frac{bx^5}{120a}-\frac{cx^5}{120a}-\tfrac{1}{8}a^3bx-\tfrac{5}{24}a^3cx.$$

No constant being added.

When $x=a$, $EIy=-\frac{3}{40}a^4b-\frac{2}{15}a^4c$.

If $c=0$, $EIy=-\frac{3}{40}a^4b$. If $b=c$, $EIy=-\frac{5}{24}a^4b$.

y is the deflection required in each case.

Also solved by S. G. Barton.

PROBLEMS FOR SOLUTION.

ALGEBRA.

333. **Proposed by R. D. CARMICHAEL, Princeton University.**

Sum the infinite series

$$\frac{1}{(m+1)^2}+\frac{(2m-1)}{(2m+1)^2}+\frac{(3m-1)^2}{(3m+1)^4}+\frac{(4m-1)^3}{(4m+1)^5}+\frac{(5m-1)^4}{(5m+1)^6}+\ldots$$

334. **Proposed by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.**

Sum the series, $2^n-n\,2^{n-2}+\frac{n(n-3)}{2!}2^{n-4}-\frac{n(n-4)(n-5)}{3!}2^{n-6}$

$$+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-8}-\frac{n(n-6)(n-7)(n-8)(n-9)}{5!}2^{n-10}+\ldots$$

GEOMETRY.

362. **Proposed by V. M. SPUNAR, M. and E. E., 3536 Massachusetts Avenue, N. S., Pittsburg, Pa.**

Show that the focus of an ellipse may be regarded as an indefinitely small circle having double contact with the ellipse, the directrix being the chord joining the points of contact.

363. **Proposed by G. I. HOPKINS, Manchester, N. H.**

Construct the triangle, having given, base, vertical angle, and difference between altitude and sum of the other two sides.

364. **Proposed by R. C. ARCHIBALD, Providence, R. I.**

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. [''Euclidean constructions'' are particularly desired.]

CALCULUS.

290. **Proposed by C. N. SCHMALL, New York City.**

When the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, represents an ellipse, show (by integration) that its area is

$$\frac{\pi\,(af^2+bg^2+ch^2-abc-2fgh)}{(ab-h^2)^{\frac{3}{2}}}.$$

291. **Proposed by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.**

Integrate $\frac{dy}{dx}=ay^2+bx^m$.

MECHANICS.

244. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

A load P is supported by three strings of equal size lying in the same plane. The middle string is vertical, one string makes with it the angle θ on one side, and the second string makes with it the angle ϕ on the other side. Find the stresses in the strings.

245. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A body moves with constant velocity in the circumference of an ellipse. Find the rate of approach (1) to the center, (2) to one of the foci, for any point in the ellipse.

NOTES AND NEWS.

It is desired to include in this column notes of interest concerning the activities of teachers of college mathematics in the field of education. The editors will gladly receive news of this kind from any source. Such items may be transmitted to any one of the three editors.

Columbia University announces that several new officers of instruction will be added to the Department of Mathematics at the close of the present academic year.

Professor S. C. Newson, of the Department of Mathematics in the University of Kansas, died suddenly at his home on Thursday evening, February 17, 1910.

Professor Granville's paper in the present number is the second in the series on the teaching of college mathematics. The next paper in this series will be by Professor H. L. Rietz, of the University of Illinois, on the Teaching of College Algebra.

A series of five lectures on graphical methods in the teaching of mathematics will be given at the University of Michigan during February, by Professor Carl Runge of Goettingen, Germany, who is Kaiser Wilhelm Exchange Professor at Columbia University during the present year. The titles include: Methods of graphical calculation; Graphical representation of functions, Graphical integration and differentiation, Graphical treatment of differential equations.

Mr. R. M. Ginnings, formerly of Kirksville Normal School, Missouri, is professor of mathematics in the State Normal School at Macomb, Illinois. Mr. Ginnings had recently completed a year's graduate study, on leave of absence from Kirksville, and had attained the Master's degree in mathematics.

Professor R. L. Rhoton of Georgetown College, Kentucky, is chairman of a committee of the State Teachers' Association to revise the course of study in mathematics in the Schools of Kentucky.

Mr. E. E. Whitford, Tutor in Mathematics in the College of the City of New York, is the author of an article in the January number of *School Science and Mathematics* entitled: "An American Syllabus in Algebra," in which the results of the various algebra syllabi of recent years are summed up in tabular form for convenient reference.

Professor W. A. Manning, of Stanford University, is exchange professor at the University of Illinois during the present year.

Professor H. L. McAlister, of Ouachita College, was a member of the committee of five of the State Teachers' Association of Arkansas, by whom was prepared the recently published outline of a four years' course in high school mathematics. The algebra section of this outline is a reproduction of the report on algebra published by the Central Association of Science and Mathematics in 1907, which has had such a wide circulation and influence during the past three years.

BOOKS.

Rara Arithmetica. A catalog of the Arithmetics written before the year MDCI, with a description of those in the Library of George Arthur Plimpton of New York. By David Eugene Smith, of Teachers' College, Columbia University. Large 8vo. Cloth. xiii+507 pages. Price, $5.50. Boston and Chicago: Ginn & Co.

This work is of great value to the teacher of elementary mathematics, and should be eagerly sought after by those who are teaching arithmetic. One needs to know the evolution of every subject he is teaching, and never before has there been given such an exhaustive description of the arithmetics used prior to 1601.

The book abounds with facsimile reproductions of title pages and figures.

The publishers have spared no pains to make the book neat and attractive. Their undertaking in the production of books that are of inestimable value to the scholar and investigator, but of little commercial value, should be encouraged by a large demand for such books as the one before us. F.

Ganot's Physics. Seventeenth Edition, Revised to 1905. By A. W. Reinhold, M. A., F. R. S., Professor of Physics, Royal Naval College, Greenwich, England. 8vo. Cloth, xi+1169 pages. Price, $5.00. New York: William Wood & Co.

In the present edition, while the general character and scope of the work, which marked the Sixteenth Edition have been maintained, considerable additions and modifications in detail have been made, and the whole has been carefully revised.

The chapter on the Steam Engine has been rewritten and brought up-to-date; a similar service has also been performed in connection with the articles on Photography by competent authorities.

Numerous additional examples and problems especially in Magnetism and Electricity will be found at the end of the volume.

Professors of some of the largest universities in America, consider it the best all around text-book on this subject published. PUBLISHERS.

The typogoaphy of this book is very good, and its engravings have been drawn upon by many of the foremost authors who have written texts on Physics since the appearance of the first edition of Ganot's. The seventeenth edition is a splendid work and deserves to be popular. F.

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THE TEACHING OF COLLEGE ALGEBRA.

By H. L. RIETZ, University of Illinois.

1. Introduction. The expression "College Algebra" is used in this paper to designate the algebra which should be taught to freshmen students who have met the entrance requirements of one and one half years of algebra in the secondary schools. This expression is here employed because it has come into rather general use in this country for the course given in the first college year, and not with any thought that it is deemed adequate algebraic training to be expected from a college graduate interested in mathematics. On this point, it is to be understood that a much more advanced course, which includes such subjects as the theory of linear dependence and the theory of matrices, should be taken late in the college course.

The question of the time at our disposal may considerably modify the character of the course in college algebra. In what follows, the writer assumes, as a minimum of time available for college algebra, that time which should be apportioned to algebra if trigonometry, algebra, and analytic geometry together are to constitute a course with five class periods per week through a college year.

2. Selection of material. The chief danger in the selection of material and presentation of college algebra is that it is likely to be a sort of scrap heap of disconnected or rather remotely connected topics, rather than an organized body of knowledge. While this is a danger, it is not a necessary characteristic of the course. The number of subjects classed under college algebra is usually too large for the time allotted to the course. The presentation of so many subjects is due, in the main, to the call from branches of mathematics higher up for a working knowledge on the part of the student of various algebraic methods; and, certainly, one criterion for judging the importance of a subject is its use in other branches of mathematics. It is not undesirable to treat a large number of topics, provided the time is available, and if there are a few underlying principles or a method of attack by which the topics are unified or brought into relation with each other. In fact, it is familiarity with a large variety of topics and their inter-relations that should be sought in elementary college courses in mathematics.

3. Unifying elements of the course. It appears that there are at least two ways by which topics of a branch of mathematics may be educationally unified; that is, so presented or organized that the student feels that the branch in question is a connected body of knowledge. For example, in the study of Euclid, a student is surely impressed with the logical element. That is to say, the body of geometric facts seems to him to be unified by definitions and axioms. In the study of analytic geometry, the student is hardly impressed with logical considerations at all, but the subject should be as thoroughly unified in the method of presentation by reference to the idea of correspondence between an equation and a curve, as is Euclid by logical considerations.

College algebra should be unified, to some extent, by logical considerations; but, in the opinion of the writer, it should be presented so that the student will feel that it is even more unified by reference to a few central facts, as in the case of analytic geometry. While secondary school algebra should perhaps be presented, in the main, by generalizations from particular numerical cases, in college algebra, the explicit statements of definitions and assumptions on which to base proofs, and the deduction from these statements of some principles of algebra, should constitute one kind of unifying basis. This does not imply, by any means, that the course includes a critical study of fundamentals. Freshmen students are not prepared for such a study, but they are prepared to appreciate that principles of algebra, as well as the propositions of geometry, can be established by logical arguments. The number of stated assumptions should be too many rather than too few, provided the assumptions are readily accepted by the student and are mathematically sound.

In §§7—13 of this paper, it is the purpose to indicate briefly how some of the subjects of college algebra, that are likely to appear isolated, can be unified in method of presentation by much reference to the *equation as a condition to be satisfied,* and to the *function as a variable whose changes are to be traced.*

4. Relation to secondary algebra. The view has been frequently expressed by writers on mathematical education that the mathematics of the college is not well correlated with the mathematics as taught at present in the secondary schools, and that the ideas of the best trained mathematicians have little influence on secondary instruction. To meet the need thus felt for a closer correlation of ideas, one of the aims of the course in college algebra should be to bridge the gap between the ideas emphasized in the high school and in the college by treating topics of secondary algebra in the college course more in the light of higher mathematics, and with greater stress on logical considerations than is, in general, feasible in the high school.

5. Nature of exercises and problems. In connection with the establishment of each important principle, there should be given, whenever feasible, illustrative problems so connected with the experience of the pupil as

to make the principle appear of real value on account of its applications. That such problems exist has been shown, but the teacher can do a great service by adding to the supply existent.

6. A high school view of equations. Almost invariably, the student who has completed merely the algebra offered in our secondary schools, regards the equation solely as an equality containing unknowns to be found. To find these unknowns, the student has learned to perform certain mechanical processes on the members of the equation — these processes being suggested by the simple operations of arithmetic and not really established for the more general numbers of algebra. The validity of the processes under any conditions is hardly ever called into question in the secondary algebra. Whether substantial improvement in results bearing on this point are to be expected from secondary schools in the near future is beyond the purpose of this paper. But, at present, surely, the condition obtains that in so far as the equation is an object of thought for the entering freshman, he regards it almost solely with reference to finding the unknowns it contains.

7. A second view of equations. In college algebra, the chief emphasis should be placed on the equation as a *condition* to be satisfied rather than as something containing unknowns to be found. To show that equations are not thus regarded by students entering college, and that this fact leads to some difficulties, let me say that I find every year in discussing the factor theorem that some of the superior students argue as follows:

"If a is a root of $f(x)=0$, this means that the unknown x is a, and to say that $f(x)$ is divisible by $x-a$ is equivalent to saying that zero is a divisor of $f(a)$. But this is contrary to the fundamental statement that division by zero is excluded from mathematical operations."

To lead the student to correct this fallacy, and to substitute for it correct views on such points is one of the difficult tasks of the instructor in college algebra.

8. Idea of functionality. In order to appreciate fully and deeply the significance of satisfying even simple equations such as $y=3x+4$ and $y=4x-6$, the graph is almost indispensable. With the graph comes the idea of functionality; and, in my opinion, a treatment of the graphs of linear and quadratic functions should precede the review of the linear and quadratic equations in this course.

To get some idea of the recent tendency to appeal to geometrical representations in college algebra, I have just examined eight text books in college algebra that I happen to have on hand — four of these books have been written within the past five years, and four others from ten to nineteen years ago. I find that the newer set of books are adorned with curves and figures to the extent of from twenty-four to thirty-nine cuts while the older books contain from none to nineteen cuts — most of the latter being given to represent complex numbers. While I recognize that the character of the text is not, in general, identical with the course, I think the change in the

character of text books is a significant measure of the tendency to make graphs and the idea of functionality more prominent than formerly.

9. Determinants. In expressing solutions of linear equations, and in placing conditions on coefficients of various equations to correspond to certain facts in regard to the roots, the determinant notation is such an important mechanical aid in giving elegance of form to the work that the elements of determinants should be presented for use in this course and in analytical geometry, rather than as an isolated subject almost entirely separated from the equation.

10. Extensions of the number concept. In seeking numbers to satisfy equations, there is the opportunity to interest the student in extensions of the number concept, and to treat in a very elementary way irrational and complex numbers. The writer has been much impressed by the interest of his students in making extensions of the number concept appear necessary to meet the demands of the equation. This method of extending the number concept may, of course, give a wrong view unless it is pointed out that some irrational numbers, such as π for example, are not roots of an equation with rational coefficients.

11. Theory of equations. In the time at our disposal, we can perhaps expect only that the solution for real roots of special numerical equations be reached as a goal in the theory of equations. This problem is not only of value for its direct applications, but also because it brings before the student a process of successive approximation of much importance as an illustration of general methods of successive approximation. However, the points to be kept most prominently before the student in the theory of equations are that we are regarding $f(x)=0$ as a relation to be satisfied, and $f(x)$ as a function whose changes in value concern us.

11. Logarithms. In the plotting of graphs of functions, expressions of sufficient complexity should be given to bring the student to seek the best methods of numerical evaluation. This calls for the use of logarithms, which, it is generally hoped, are used in calculations in the secondary course, but are often not used in such a course.

12. Limits and infinite series. There seems to exist considerable difference of opinion among instructors of college mathematics as to the advisability of presenting the elements of the theory of limits and infinite series in a freshman course. If the subject of infinite series is to be included in the course, it can be very effectively presented with emphasis on the fact that the first n terms of a series is a function which changes as n takes values 1, 2, 3, 4, ..., and that it is the limit of this function as n grows beyond any fixed bound in which we are interested. In this way, the subject of infinite series is referred to one of the unifying elements of the course.

To write the generating function from a few terms of a series is a most valuable exercise in leading students to grasp what may be called algebraic form. Herein is the recognition of algebraic law from particular

cases. Even the simple direct process of expanding $\sum_{n=1}^{n=\infty} u_n$, where u_n is some function of n such as $\frac{1}{(2n-1)\,!}$, forms a sort of prelude to the use of the summation sign in the calculus.

The value of the subject of infinite series does not therefore rest solely either on the practical ground that the student needs to test the convergence of special series that arise in analysis, nor on the ground that it is a suitable field for the growth of the limit concept.

The writer does not find himself in agreement with those who hold that it is a more natural order of presentation to defer the study of infinite series until late in the course in calculus. He questions whether this position is tenable, especially if series are presented with a considerable use of geometric representation. That is to say, if the sums of n terms of a few carefully selected series are plotted as ordinates to correspond to $n=1, 2, 3, \ldots$ used as abscissas, so that the function idea is prominent, the process of approach to a limit makes a valuable appeal to the student's common sense. While the answer to the question of including this subject in a freshman course may be changed by the amount of time allotted to the course in calculus, it is the opinion of the writer that a good deal is gained by presenting to freshmen students in considerable detail, the elementary tests for the convergence and divergence of series, if fewer than five hours per week through a school year are given to calculus.

Wherever convergence of series is studied, the treatment should be marked by precision of statements. The arguments should be in harmony with the unifying principles of this course in algebra; that is, assumptions on which proofs are based should be explicitly stated.

13. Summary. In this paper, the writer has aimed to consider especially certain subjects whose place in college algebra is sometimes called into question, and to direct attention to points which should be particularly emphasized.

It seems that while the course in college algebra should be unified, to some extent, by logical considerations, it should be unified in method of presentation by much reference to 1) the equation as a relation to be satisfied, 2) the function as a variable whose changes in value are to be traced.

It is by referring much to these central and closely related ideas, and by the introduction of more practical problems (§ 5) that we may well expect to improve our teaching of college algebra.

ON r-FOLD SYMMETRY OF PLANE ALGEBRAIC CURVES.*

By R. D. CARMICHAEL, Princeton, N. J.

If a plane curve is revolved about a point in its own plane through an angle of $360°/r$ and if it then coincides with its former position, it is said to have r-fold symmetry with respect to the point; and the point is called the center of r-fold symmetry. The object of this paper is to ascertain the analytical conditions which are necessary and sufficient to the existence of r-fold symmetry and to examine into the geometric properties of the curves in certain special cases. In a previous note† I have given a classification of plane algebraic curves having four-fold symmetry about a point, and this has been followed‡ by a paper on the geometric properties of quartic curves possessing such four-fold symmetry.

In the present discussion we shall confine ourselves to plain algebraic loci which are such that no locus is composed entirely of isolated points or of straight lines; in other words, every locus considered will be assumed to have at least one part which is *continuous* and *curved.* And this assumption is made throughout without further statement.

Evidently the circle is a curve of infinite-fold symmetry. It is clear that the condition of infinite-fold symmetry with respect to the origin is that the polar equation shall be independent of the vectorial angle; that is, the locus in this case is a circle or a system of concentric circles with center at the center of infinite-fold symmetry. Therefore it will be sufficient in what follows to confine our attention to the cases in which r is finite.

1. *Separation into two classes.* Let n be the order of a curve of r-fold symmetry and let it be referred to rectangular cartesian coordinates with origin at the center of r-fold symmetry. Take the equation in the form

$$\Sigma\, a_{ts}x^t y^s=0, \tag{1}$$

where a_{ts} is a real constant for every t and s and where t and s each range over the values 0, 1, 2, ..., n subject to the condition

$$t+s \leqq n.$$

Certain relations must exist among the coefficients a. These are now to be found.

If we transform equation (1) by the substitution

*Presented to the American Mathematical Society, December 30, 1908.

†*Annals of Mathematics,* Vol. 9, No. 2, pp. 53-56.

‡*Annals of Mathematics,* Vol. 10, No. 2, pp. 81-87.

$$x=\rho \cos \theta, \quad y=\rho \sin \theta,$$

we have

(2) $$\sum_{v=0}^{v=n} \rho^v \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t \theta \sin^s \theta=0, \quad (t+s=\nu).$$

Putting $\theta=\theta_1$ and $\theta=\theta_1+\alpha\,\phi$ where

$$\phi=360^\circ/r \text{ and } \alpha=\text{an integer},$$

we have the following equations:

(3) $$\sum_{v=0}^{v=n} \rho^v \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1=0, \quad (t+s=\nu),$$

(4) $$\sum_{v=0}^{v=n} \rho^v \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t (\theta_1+\alpha\,\phi) \sin^s (\theta+\alpha\,\phi)=0, \quad (t+s=\nu).$$

From the existence of the defined r-fold symmetry it follows that equations (3) and (4) must yield by solution the same values of ρ. Therefore the coefficients can differ only by a constant factor m_α; that is,

(5) $$\sum_{t,\, s=0}^{t;\, s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1=m_\alpha \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t (\theta_1+\alpha\,\phi) \sin^s (\theta_1+\alpha\,\phi), \quad (t+s=\nu).$$

Equation (5) must hold for each value of ν from 0 to n and for each value of α from 1 to r, a different equation being formed for every case. Then (5) yields $r(n+1)$ equations which must all be satisfied for every possible value of θ_1. It is clear that the existence of this system of equations is both necessary and sufficient to the existence of the defined symmetry.

We shall now evaluate the constants m_α. If we take $\alpha=1$ and multiply equation (4) by m_1 (which evidently cannot be zero) it follows that the result is identical with equation (3). Hence a second addition of ϕ to the vectorial angle would necessitate a second multiplication of the coefficients by m_1; that is, two multiplications by m_1 produces m_2; or

$$m_2=m_1^2.$$

Continuing the additions of ϕ to the vectorial angle, we have

$$m_2=m_1^2, \quad m_3=m_1^3, \quad \ldots, \quad m_r=m_1^r.$$

At the rth addition of ϕ to the vectorial angle the two members of (5) become identical except as to the presence of the factor m_r in the second member; and therefore m_r must equal 1. For, if not, we must have

$$\Sigma\, a_{ts} \cos^t \theta_1 \sin^s \theta_1 = 0, \quad (t+s=\nu),$$

for an unlimited number of values of θ_1 each less than 2π; and this is evidently impossible. Since $m_r = m_1{}^r$ and $m_r = 1$, we have $m_1{}^r = 1$. It is evident from (5) that m_1 is a real quantity. Therefore

(6) $\quad m_1 = +1$, when r is odd;
(7) $\quad m_1 = \pm 1$, when r is even;
(8) $\quad m_a = m_1{}^a$, in every case.

In order to find the necessary and sufficient relations among the coefficients a we proceed as follows. For $m_1 = +1$, $m_1 = -1$, equation (5) takes the respective forms:

$$(9) \quad \sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t\theta_1 \sin^s\theta_1 = \sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t(\theta_1 + a\,\phi) \sin^s(\theta_1 + a\,\phi), \quad (t+s=\nu), \quad [A],$$

$$(10) \quad \sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t\theta_1 \sin^s\theta_1 = (-1)^a \sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t(\theta_1 + a\,\phi) \sin^s(\theta_1 + a\,\phi), \quad (t+s=\nu), \quad [B],$$

where ν ranges over all the values 0, 1, 2, ..., n, different equations being formed for each value of ν. Equation (9) alone holds when r is odd; when r is even both equations (9) and (10) may hold. Evidently these equations are necessary and sufficient to the existence of r-fold symmetry; that is, for odd-fold symmetry we must be able to satisfy (9); for even-fold symmetry, either (9) or (10) or both. In case this condition cannot be satisfied for given r and n, we are to conclude that r-fold symmetry does not exist for curves of such degree n. As a case in point, we have the theorem: Four-fold symmetry does not exist for curves of odd degree.*

We shall say that curves which satisfy equations (9) and (10) are of class A and B, respectively. In class A there will be found loci of both odd- and even-fold symmetry; in class B will be found loci of only even-fold symmetry.

Obviously, if the equation of a curve referred to rectangular axes has only terms of even degree or only terms of odd degree, the curve has two-fold symmetry; for in either case, if α, β is a point on the curve, so is $-\alpha$,

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$-\beta$. Conversely, if the origin of rectangular coordinates is taken at the center of two-fold symmetry, the equation must evidently have one of the forms indicated. If each term is of even degree, it is obvious that the curve belongs to class A; while if each term is of odd degree, the curve belongs to class B.

2. Determination of constants a_{ts} for class A. Equation (9) indicates that the real function

$$F \equiv \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1, \quad (t+s=\nu), \tag{11}$$

is periodic with the real period $\phi = 2\pi/r$. But every real function of a single variable θ_1 with the real period $2\pi/r$ can be expanded in a Fourier series in the general form

$$\sum_{i=0}^{\infty} c_i \cos ir\,\theta_1 + \sum_{i=1}^{\infty} y_i \sin ir\,\theta_1. \tag{12}$$

If $\cos ir\,\theta_1$ and $\sin ir\,\theta_1$ are expanded in terms of $\sin \theta_1$ and $\cos \theta_1$ the results are homogeneous of order ir in $\sin \theta_1$, $\cos \theta_1$; moreover, the coefficients c_i and y_i do not belong to terms alike in $\sin \theta_1$ and $\cos \theta_1$, and therefore cannot annul each other. Hence, if the expression in (12) is to be identical with F, $ir \leqq \nu$. If $ir = \nu - 2j$, the corresponding part of (12) when expanded in terms of $\cos \theta_1$, $\sin \theta_1$ is of degree $\nu - 2j$; but it becomes of degree ν through multiplication by the unit factor $(\cos^2 \theta_1 + \sin^2 \theta_1)^j$. Evidently ir cannot differ from ν by an odd number. Hence, as a result we have

$$\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{i=0}^{i} c_i \cos ir\,\theta_1 + \sum_{i=1}^{i} y_i \sin ir\,\theta_1, \tag{13}$$

where $t+s=\nu$ and ir is always positive and has as its values some or all of the positive numbers of the series ν, $\nu-2$, $\nu-4$, ...

Now, if ν ranges from 0 to n, the preceding result enables us to determine readily the values of all of the coefficients a_{ts} in terms of a suitable number of them selected as independent constants. Substituting these values in (1) we obtain the most general form of the equation of the nth degree locus possessing r-fold symmetry and belonging to class A as defined above. Such equations, for several values of r and n, are written out below in their most general form.*

*For curves of four-fold symmetry see my previous papers already referred to. Curves of two-fold symmetry are disposed of at the close of section 1 of this paper.

SOME CURVES OF CLASS A OF 3-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$$
$$F_3 \equiv F_2 + c_3 x^3 + c_4 y^3 - 3c_4 x^2 y - 3c_3 xy^2 = 0.$$
$$F_4 \equiv F_3 + c_5 (x^2 + y^2)^2 = 0.$$
$$F_5 \equiv F_4 + c_6 x^5 + c_7 y^5 - 3c_7 x^4 y - 3c_6 xy^4 - 2c_6 x^3 y^2 - 2c_7 x^2 y^3 = 0.$$
$$F_6 \equiv F_5 + c_8 (x^6 - y^6) + 6c_9 xy(x^4 + y^4) - 15c_8 x^2 y^2 (x^2 - y^2) \\ - 20c_9 x^3 y^3 + c_{10} (x^2 + y^2)^3 = 0.$$
$$F_7 \equiv F_6 + (x^2 + y^2)(c_{11} x^5 + c_{12} y^5 - 3c_{12} x^4 y - 2c_{11} x^3 y^2 - 2c_{12} x^2 y^3) = 0.$$

SOME CURVES OF CLASS A OF 5-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$$
$$F_4 \equiv F_2 + c_3 (x^2 + y^2)^2 = 0.$$
$$F_5 \equiv F_4 + c_4 x^5 + c_5 y^5 + 5c_5 x^4 y + 5c_4 xy^4 - 10c_4 x^3 y^2 - 10c_5 x^2 y^3 = 0.$$
$$F_6 \equiv F_5 + c_6 (x^2 + y^2)^3 = 0.$$
$$F_7 \equiv F_6 + (x^2 + y^2)(c_7 x^5 + c_8 y^5 + 5c_8 x^4 y + 5c_7 xy^4 \\ - 10c_7 x^3 y^2 - 10c_8 x^2 y^3 = 0.$$

SOME CURVES OF CLASS A OF 6-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$$
$$F_4 \equiv F_2 + c_3 (x^2 + y^2)^2 = 0.$$
$$F_6 \equiv F_4 + c_4 (x^2 + y^2)^3 + c_5 (x^6 - y^6) + 6c_6 xy(x^4 + y^4) - 15c_5 x^2 y^2 (x^2 - y^2) \\ - 20c_6 x^3 y^3 = 0.$$

SOME CURVES OF CLASS A OF 7-FOLD SYMMETRY.

$$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$$
$$F_4 \equiv F_2 + c_3 (x^2 + y^2)^2 = 0.$$
$$F_6 \equiv F_4 + c_4 (x^2 + y^2)^3 = 0.$$
$$F_7 \equiv F_6 + c_5 x^7 + c_6 y^7 - 7c_6 x^6 y - 7c_5 xy^6 - 21c_5 x^5 y^2 - 21c_6 x^2 y^5 \\ + 35c_6 x^4 y^3 + 35c_5 x^3 y^4 = 0.$$

3. Determination of the constants a_{ts} for class B. For class B we have seen that r is even. It may be shown that r-fold symmetry in class B is a special case of $\frac{1}{2}r$-fold symmetry in class A. For if ϕ is the angle through which the r-fold (r even) symmetrical curve of class B must be turned in order to coincide with its original position, 2ϕ is the angle through which the $\frac{1}{2}r$-fold symmetrical curve of class A must be turned that it may coincide with its original position. But the former still coincides with its first posi-

tion after being turned through an angle of 2ϕ. Hence, since r must be even for curves of class B, r-fold symmetry of class B is a special case of $\frac{1}{2}r$-fold symmetry of class A.

For $\frac{1}{2}r$-fold symmetry equation (13) becomes

$$\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts}\cos^t\theta_1\sin^s\theta_1=\sum_{i=0}^{i} c_i\cos\frac{ir}{2}\theta_1+\sum_{i=1}^{i} y_i\sin\frac{ir}{2}\theta_1, \tag{14}$$

where $t+s=\nu$ and $ir/2$ is always positive and has as its values some or all the positive numbers of the series ν, $\nu-2$, $\nu-4$, ... This is a necessary condition for r-fold symmetry in class B. From (10) we may write

$$\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts}\cos^t\theta_1\sin^s\theta_1=-\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts}\cos^t(\theta_1+\phi)\sin^s(\theta_1+\phi),\quad (t+s=\nu). \tag{15}$$

Since the existence of equation (10) is a necessary and sufficient condition for r-fold symmetry of curves of class B, it is readily seen from the discussion in the preceding paragraph that the existence at the same time of equations (14) and (15) is also a necessary and sufficient condition for r-fold symmetry of curves of class B. This result enables one to determine the constants a_{ts} in terms of a suitable number of them chosen as independent constants.

But if r is twice an odd number, the constants may be more readily determined in the following manner: In the equation for the curve of class A of $\frac{1}{2}r$-fold symmetry, insert the condition for two-fold symmetry of class B; that is, let the equation consist only of terms of odd degree. In this way were found the equations for 6- and 10-fold symmetry given below.

Some Curves of Class B of 6-fold Symmetry.

$F_3\equiv c_1x^3+c_2y^3-3c_2x^2y-3c_1xy^2=0.$

$F_5\equiv F_3+c_3x^5+c_4y^5-3c_4x^4y-3c_3xy^4-2c_3x^3y^2-2c_4x^2y^3=0.$

$F_7\equiv F_5+(x^2+y^2)(c_5x^5+c_6y^5-3c_6x^4y-3c_5xy^4-2c_5x^3y^2-2c_6x^2y^3)$
$\qquad+(x^2+y^2)^2(c_7x^3+c_8y^3-3c_8x^2y-3c_7xy^2)=0.$

Some Curves of Class B of 10-fold Symmetry.

$F_5\equiv c_1x^5+c_2y^5+5c_2x^4y+5c_1xy^4-10c_1x^3y^2-10c_2x^2y^3=0.$

$F_7\equiv F_5+(x^2+y^2)(c_3x^5+c_4y^5+5c_4x^4y+5c_3xy^4-10c_3x^3y^2$
$\qquad-10c_4x^2y^3)=0.$

$F_9\equiv F_7+(x^2+y^2)^2(c_5x^5+c_6y^5+5c_6x^4y+5c_5xy^4-10c_5x^3y^2$
$\qquad-10c_6x^2y^3)=0.$

A different method, however, is necessary for curves of 8-fold symmetry. In this case we must employ equations (14) and (15); or, what is the same thing, a corollary from my first paper on four-fold symmetry (already referred to) and equation (15): namely, the necessary and sufficient condition for four-fold symmetry of curves of class A is that every term in the equation shall be of even degree and that

$$a_{st}=(-1)^t a_{ts}.$$

(This result is not explicitly stated there, but is easily deduced as a corollary from the argument.) This enables us in the present case to write (15) in different form. We replace ϕ by its value 45°.

$$(16)\quad \sum_{t,\,s=0}^{t,\,s=\nu} [a_{ts}(\cos^t \theta_1 \sin^s \theta_1 + (-1)^t \cos^s \theta_1 \sin^t \theta_1)]$$

$$= -\sum_{t,\,s=0}^{t,\,s=\nu} \{a_{ts}[\cos^t(\theta_1+45°)\sin^s(\theta_1+45°)+(-1)^t\cos^s(\theta_1+45°)\sin^t(\theta_1+45°)]\},$$

where $t+s=\nu$, ν being an even number and $t \gtreqless s$. It follows that the existence of equation (16) is the necessary and sufficient condition for 8-fold symmetrical curves of class B. By its aid one may determine the equations of 8-fold symmetrical loci.

4. A simplification in constructing the equations in general. If

$$r=p_1^{a_1}\, p_2^{a_2} \ldots p_k^{a_k},$$

where $p_1, p_2, \ldots, p_k$ are different primes, we may evidently proceed as follows to construct the equations of nth degree loci possessing r-fold symmetry:

Construct the equations of nth degree loci possessing symmetry of class A and of orders $p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}$, respectively. From these construct the most general equation in which the coefficients obey all the limitations imposed in the several equations separately. The result is the most general form of the equation of class A. Proceed similarly for class B.

5. *An example.* As an illustrative example consider a special case of the seventh degree curve of class B of 10-fold symmetry in the table above. Let $c_1=c_4=0$, $c_2\neq 0$, $c_3\neq 0$. Then the equation is of the form:

$$(17)\ a_1(5x^4y-10x^2y^3+y^5)+a_2(x^2+y^2)(x^5-10x^3y^2+5xy^4)=0,\ a_1\neq 0,\ a_2\neq 0.$$

Transforming to polar coordinates by the substitution $x=\rho\cos\theta$, $y=\rho\sin\theta$, and substituting $\cos 5\theta$ and $\sin 5\theta$ for $\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ and $5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$, respectively, the equation becomes

$$a_1\rho^5\sin 5\,\theta+a_2\rho^7\cos 5\,\theta=0.$$

Evidently this may be replaced by the two equations

$$\rho^5=0,$$

(18) $$\rho^2=a\tan 5\,\theta,$$

where $$a=-a_1/a_2.$$

To the former of these two equations corresponds only the origin. But this point is on the locus of the other equation. Hence, so far as plotting the curve is concerned, equation (17) may be replaced by equation (18). Evidently, it consists of five branches, alike except for position. Each branch passes through the origin and has in itself two-fold symmetry with respect to the origin. Moreover, the origin is obviously a point of inflection for each branch, and there are thus five (but only five) points of inflection at the origin. Now, since the curve possesses 10-fold symmetry, singularities not at the origin can enter only by tens. Hence the number of points of inflection is an odd multiple of 5. It is easy to see that there is no cusp at the origin. Hence cusps enter only in tens, if at all.

For the further discussion of singularities we require the following Plücker equations, which are written in the ordinary notation:

(19) $$m=n(n-1)-(2\,\delta+3\,\rho),$$

(20) $$n=m(m-1)-(2\,\tau+3\,\iota),$$

(21) $$\iota=3n(n-2)-(6\,\delta+8\,\rho),$$

(22) $$\rho=3m(m-2)-(6\,\tau+8\,\iota).$$

We now have $n=7$, ι=odd multiple of 5, ρ=multiple of 10, or zero. Then from (21) it may be seen that $6\,\delta+8\,\rho$ must be an even multiple of 5; that is, a multiple of 10. But ρ is a multiple of 10, or zero; hence δ is a multiple of 5. It is obvious from (20) that $m \overline{>} 4$; hence from (19) it follows that either δ or ρ is zero; and therefore $\rho=0$, since the curve under consideration has double points at the origin. Now the locus is of the seventh degree and cannot have as many as ten coincident points; hence, since δ is a multiple of 5, the number of double points at the origin is 5. Therefore, from (19) it follows that $\delta=5$ or 15, since singularities not at the origin enter only by tens. We shall now determine which of these is the true value.

Suppose that there is a double point not at the origin; and let it be at a distance d from the origin. Then there must be ten such double points at a distance d from the origin. Pass through them a circle with radius d

and center at the origin. Since each double point counts as two points the circle cuts the septic curve in 20 points. But this is impossible, hence there is no double point except at the origin. Therefore $\delta=5$. Then from Plücker's equations: $m=32$, $\iota=75$, $\tau=380$. Hence the curve

$$5x^4y-10x^2y^3+y^5+c(x^2+y^2)(x^5-10x^3y+5xy^4)=0,\ c\neq 0.$$

is of class 32, is non-cuspidal, and has five double points at the origin, 75 points of inflection of which five are at the origin, and 380 double tangents. It is obvious that not all the singularities are real.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n\doteq\infty} x^{1/(2n-1)},\ n=1,\,2,\,\ldots;\quad f(x)=\lim_{n\doteq\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

No solution of this problem has been received.

331. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Extract the square root of $21+6\sqrt{2}+2\sqrt{21}-6\sqrt{3}-6\sqrt{7}-2\sqrt{6}-2\sqrt{14}$ and also of $4\sqrt{2}+2\sqrt{6}-9-4\sqrt{3}$.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y., and J. SCHEFFER, A.M., Hagerstown, Md.

(a) Assume the root to be of the form

$$a\sqrt{2}+b\sqrt{3}+c\sqrt{7}+d.$$

Squaring and comparing coefficients, we have

$$ab=-1,\ ac=-1,\ ad=3,\ bc=1,\ bd=-3,\ cd=-3,$$
$$2a^2+3b^2+7c^2+d^2=21.$$

Whence $a=-1$, $b=1$, $c=1$, $d=-3$, and the root is $\sqrt{3}-\sqrt{2}+\sqrt{7}-3$.

For second expression, (b), assume root of form $\sqrt{-1}(a+b\sqrt{3}+c\sqrt{2})$. Squaring and comparing coefficients, we have,

$$ab=2,\ ac=-2,\ bc=-1,\ a^2+3b^2+2c^2=9.$$

Whence $a=2$, $b=1$, $c=-1$, and the root is $\sqrt{-1}(2+\sqrt{3}-\sqrt{2})$.

Solved similarly by **V. M. Spunar** and **Levi S. Shively**.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

The method used here was suggested to me by Mr. Githens in a solution for cube root. It is considerably simpler than methods ordinarily used.

(a) Let $A=[21+6\sqrt{2}+2\sqrt{21}-(6\sqrt{3}+6\sqrt{7}+2\sqrt{6}+2\sqrt{14})]^{\frac{1}{2}}$
$=\sqrt{(m-n)}$.

Then $\sqrt{[(m-n)(m+n)]}=[157+12\sqrt{2}-(4\sqrt{21}+24\sqrt{42})]^{\frac{1}{2}}=\sqrt{(p-q)}$.

$\sqrt{[(p-q)(p+q)]}=(409-264\sqrt{2})^{\frac{1}{2}}=\sqrt{(r-s)}$.

$\sqrt{[(r-s)(r+s)]}=167$.

Let $\sqrt{(r-s)}=\sqrt{(x+167)}-\sqrt{x}$. Squaring both members and equating rational terms, $2x+167=409$, or $x=121$.

$\therefore \sqrt{(r-s)}=12\sqrt{2}-11$.

Let $\sqrt{(p-q)}=\sqrt{(x+12\sqrt{2}-11)}-\sqrt{x}$.

Then $2x+12\sqrt{2}-11=157+12\sqrt{2}$, or $x=84$.

$\therefore \sqrt{(p-q)}=\sqrt{(73+12\sqrt{2})}-2\sqrt{21}$.

$\sqrt{[(73+12\sqrt{2})(73-12\sqrt{2})]}=71$.

Hence $\sqrt{(73+12\sqrt{2})}=\sqrt{(x+71)}+\sqrt{x}$.

From this, $x=1$ and $\sqrt{(73+12\sqrt{2})}=6\sqrt{2}+1$.

$\sqrt{(p-q)}=6\sqrt{2}+1-2\sqrt{21}$.

Let $\sqrt{(m-n)}=\sqrt{(x+6\sqrt{2}+1-2\sqrt{21})}-\sqrt{x}$.

Then $2x+6\sqrt{2}+1-2\sqrt{21}=21+6\sqrt{2}+2\sqrt{21}$.

$\therefore x=10+2\sqrt{21}$. $\therefore \sqrt{(m-n)}=\sqrt{(11+6\sqrt{2})}-\sqrt{(10+2\sqrt{21})}$.

$$\sqrt{[(11+6\sqrt{2})(11-6\sqrt{2})]}=7,\ \text{and}\ \sqrt{(11+6\sqrt{2})}=\sqrt{(x+7)}+\sqrt{x}\ldots(1).$$
$$\sqrt{[(10+2\sqrt{21})(10-2\sqrt{21})]}=4,\ \text{and}\ \sqrt{(10+2\sqrt{21})}=\sqrt{(x+4)}+\sqrt{x}\ldots(2).$$

From (1), $x=2$; from (2), $x=3$.

$\therefore \sqrt{(11+6\sqrt{2})}=3+\sqrt{2}$, $\sqrt{(10+2\sqrt{21})}=\sqrt{7}+\sqrt{3}$.

$\therefore \sqrt{(m-n)}=3+\sqrt{2}-\sqrt{7}-\sqrt{3}$.

(b) Let $(-9-4\sqrt{3}+4\sqrt{2}+2\sqrt{6})^{\frac{1}{2}}=\sqrt{(-m+n)}$.

$\sqrt{[(-m+n)(-m-n)]}=\sqrt{[(-m)^2-n^2]}=\sqrt{[73+40\sqrt{3}]}=\sqrt{[p+q]}$.

$\sqrt{[(p+q)(p-q)]}=23$.

Let $\sqrt{(p+q)}=\sqrt{(x+23)}+\sqrt{x}$. Then $2x+23=73$, or $x=25$.

$\therefore\sqrt{(p+q)}=4\sqrt{3}+5$.

Let $\sqrt{(n-m)}=-\sqrt{(x+4\sqrt{3}+5)}+\sqrt{x}$. Then $2x+4\sqrt{3}+5=-9-4\sqrt{3}$, or $x=-7-4\sqrt{3})$.

$\therefore\sqrt{(n-m)}=\sqrt{(-7-4\sqrt{3})}-\sqrt{-2}$.

$\sqrt{[(-7-4\sqrt{3})(-7+4\sqrt{3})]}=1$.

Let $\sqrt{(-7-4\sqrt{3})}=\sqrt{(x+1)}+\sqrt{x}$; then $2x+1=-7$, $x=-4$.

$\therefore\sqrt{(-7-4\sqrt{3})}=\sqrt{(-3)}+2\sqrt{(-1)}$.

$\therefore\sqrt{(n-m)}=\sqrt{(-3)}+2\sqrt{(-1)}-\sqrt{(-2)}$.

Also solved by G. I. Hopkins and A. H. Holmes.

332. **Proposed by C. N. SCHMALL, New York City.**

Solve the quadratic, $x^2+ax+b=0$, without completing the square.

Solution by ARTEMAS MARTIN, LL. D., Washington, D. C.

Assume $y-\frac{1}{2}a=x$, and substitute in the given quadratic and it becomes

$$(y-\tfrac{1}{2}a)^2+a(y-\tfrac{1}{2}a)+b=0,\ \text{or}\ y^2-\tfrac{1}{4}a^2+b=0;$$

whence $y=\pm\sqrt{(\frac{1}{4}a^2-b)}$ and $x=\pm\sqrt{\frac{1}{4}a^2-b)}-\frac{1}{2}a$.

See *Mathematical Magazine*, Vol. I, No. 9 (January, 1884), p. 146.

Solved similarly by V. M. Spunar , Levi S. Shively and the Proposer.

Professor Hopkins, in his solution, made use of the principle that the sum of the roots is equal to the coefficient of x with sign changed, and the product of the roots is equal to the final term.

S. Lefsehetz sent in solutions of 327 and 328 too late for credit in last issue.

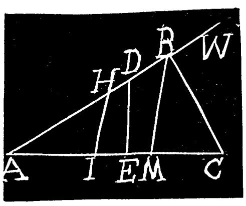
GEOMETRY.

356. **Proposed by G. I. HOPKINS, Manchester, N. H.**

Required to construct a triangle having given, base, vertical angle, and difference of other two sides.

I. **Solution by J. M. ARNOLD, Crompton, R. I.**

Let x=the longer side, then $x-d$=the shorter side. Let A=the vertical angle. Then



$$x^2-(x\cos A)^2=(x-d)^2-(b-x\cos A)^2$$

which gives

$$(2b\cos A-2d)x=b^2-d^2.$$

Dividing by 4 and putting in the form of a proportion

$$(\tfrac{1}{2}b\cos A-\tfrac{1}{2}d) : \frac{b-d}{2}=\frac{b+d}{2} : x.$$

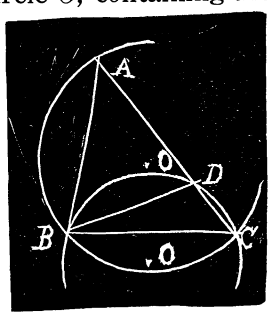
Construction. Draw AC equal to b, draw the indefinite line AW, making an angle at A equal to the given angle. On AW lay off AD equal to $\frac{1}{2}b$, and draw DE perpendicular to AC. From E lay off EI equal to $-\frac{1}{2}d$. Then will AI equal $\frac{1}{2}b\cos A-\frac{1}{2}d$.

On AW lay off AH equal to $\frac{b-d}{2}$ and draw HI. On AC lay off AM equal to $\frac{b+d}{2}$, and through M draw a line parallel to HI meeting AW at B. Join BC. Then will ABC be the triangle required.

Similarly solved by G. B. M. Zerr.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

On the given base, BC, as a chord, describe a circle O, containing the segment whose angle contains the angle$=90^\circ+\frac{1}{2}A$, A being the given vertical angle, and also a circle O', the segment of which contains the angle A. Make BD=given difference of sides; extend BD to A, where it cuts the circumference of circle O'. Draw AC; then ABC is the required triangle.



For, $\angle ADC=90^\circ-\frac{1}{2}A$.

$\therefore \angle ACD=90^\circ-\frac{1}{2}A$; $\therefore AD=AC$; $\therefore AB-AC=BD$=given difference, which proves construction.

Solved similarly C. N. Schmall and H. C. Feemster.

357. Proposed by E. R. HOYT, St. Louis, Mo.

A room is 30 feet long, 12 feet wide, and 12 feet high. At one end of the room, 3 feet from the floor, and midway from the sides, is a spider. At the other end, 9 feet from the floor, and midway from the sides, is a fly. Determine the shortest path by way of the floor, ends, sides, and ceiling, the spider can take to capture the fly.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

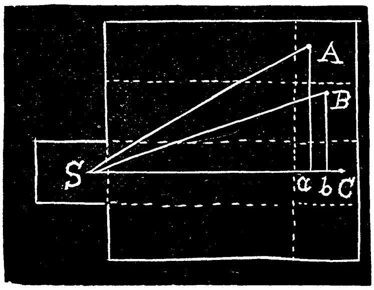
Suppose the six sides of the room spread out in one plane as in the figure, the floor being the second rectangle from the bottom, and let x=distance of spider from floor, $12-x$ the distance of the fly, $x<6$. There are three courses for the spider to take.

First, the route $SC=30+x+12-x=42$ feet... (1).

Second, the route $SB=\sqrt{[(Sb)^2+(Bb)^2]}=\sqrt{[(36+x)^2+(18-x)^2]}$... (2).

Third, the route $SA=\sqrt{[(Sa)^2+(Aa)^2]}=\sqrt{[(30+2x)^2+(24)^2]}$... (3).

Let $(30+2x)^2+576=(36+x)^2+(18-x)^2$.



Then $x^2+42x=72$, and $x=1.6495$ feet.

Let $(36+x)^2+(18-x)^2=(42)^2$.

Then $x^2+18x=72$, and $x=3.3693$ feet.

Then if $x<1.6495$ feet, the third route is shortest.

If $x>1.6495$ and <3.3693, the second route is shortest.

If $x>3.3693$, the first route is the shortest.

Since $x=3$, the spider takes the second route, and travels $\sqrt{[(39)^2+(15)^2]}=3\sqrt{[13^2+5^2)}=3\sqrt{(194)}<[3\sqrt{(196)}=3\times14=42]$ feet.

Also solved by C. N. Schmall.

358. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Neb.

Cut four coplanar non-copunctual straight lines in a harmonic range.

I. Solution by the PROPOSER.

Let a. b, c, and d be the four lines, not meeting in a point. Let a and b meet at x. Draw two lines cutting a, b, and d at A, A_1, A_2 and A, B_1, B_2, respectively. Draw A_1B_2 and A_2B_1 meeting at C_1. Draw AC_1 cutting b and d at C and C_2. Draw XC_2, and draw XC_1 cutting C at D_2. Draw C_2D_2 cutting a and b at D and D_1. $DD_1D_2C_2$ is the required range.

For ACC_1C_2 is a harmonic range determined by the four-point $A_1B_1B_2A_2$. Hence, $X-ACC_1C_2$ is a harmonic pencil. So also is $X-DD_1D_2C$; therefore DD_1D_2C is the required harmonic range.

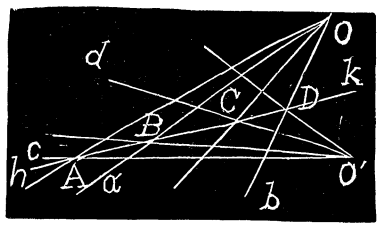
Also solved by C. N. Schmall.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let Oa, Ob, $O'c$, $O'd$ be the four lines; hk the transversal intersecting $O'c$, Oa, $O'd$, Ob in A, B, C, D, respectively.

Let $\alpha=0$ be the equation to Oa; $\beta=0$ the equation to Ob; $\gamma=0$ the equation to $O'c$; $l\alpha-m\beta+n\gamma=0$, the equation to $O'd$.

Then for a harmonic range,



$\alpha-p\beta=0$ is the equation to OC;
$\alpha+p\beta=0$ is the equation to OA;
$\gamma-q(l\alpha-m\beta+n\gamma)=0$ is the equation to $O'B$;
$\gamma+q(l\alpha-m\beta+n\gamma)=0$ is the equation to $O'D$.

$\left\{-\dfrac{2p\triangle}{b-ap}, \dfrac{2\triangle}{b-ap}, 0\right\}$, are the coordinates of A;

$\left\{\dfrac{2np\triangle}{apm+bn-clp+cm}, \dfrac{2n\triangle}{apm+bn-clp+cm}, -\dfrac{2\triangle(lp-m)}{apm+bn-clp+cm}\right\}$ are the coordinates of C;

$\left\{0, \dfrac{2\triangle(qn-1)}{b(qn-1)+cmq}, \dfrac{2\triangle mq}{b(qn-1)+cmq}\right\}$ are the coordinates of B;

$\left\{\dfrac{2\triangle(qn+1)}{a(qn+1)-qcl}, 0, -\dfrac{2\triangle ql}{a(qn+1)-qcl}\right\}$ are the coordinates of D.

$\alpha mq+\beta pmq-\gamma p(qn-1)=0$, is the line through AB;
$\alpha nql-\beta(qmn+m-lp)+\gamma n(qn+1)=0$, is the line through CD.

Comparing, we get $q=0$ or $q=-(1/n)$, $p=m/l$ or $p=0$. There are no positive values for q when p is positive. When p is positive, $q=0$, etc.

Whatever relations we establish we cannot find p and q both real or both positive.

CALCULUS.

286. Proposed by R. D. CARMICHAEL, Princeton University.

Solve the differential equation

$$[a_0x^3+a_1x^2y+a_2xy^2+(a_0-a_1+a_2)y^3 +a_3x^2+a_4xy+a_5y^2+a_6x+a_7y+a_8]dx$$
$$+[a_0y^3+a_1xy^2+a_2x^2y+(a_0-a_1+a_2)x^3 +a_3y^2+a_4xy+a_5x^2+a_6y+a_7x+a_8]dy=0.$$

No solution of this problem has been received.

287. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

An object P, being placed beyond the principal focus F of a convex lense, determine its position when its distance PQ, from its image Q, is a minimum.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let u=distance of object from lense, v=distance of image from lense, t=thickness of lense, and r, s=the radii of the first and second surface.

Then $\dfrac{1}{\dfrac{1}{v}+\dfrac{\mu-1}{s}}+\dfrac{1}{\dfrac{1}{u}-\dfrac{\mu-1}{r}}=\dfrac{t}{\mu}$, where μ=index of refraction, or

$$\mu\left(\frac{1}{u}+\frac{1}{v}\right)=\left(\frac{1}{r}-\frac{1}{s}\right)\mu(\mu-1)+t\left(\frac{1}{v}+\frac{\mu-1}{s}\right)\left(\frac{1}{u}-\frac{\mu-1}{r}\right)\ldots(1).$$

$$u+v+t=\text{minimum}\ldots(2).$$

$$\mu\left(\frac{du}{u^2}+\frac{dv}{v^2}\right)=t\left(\frac{1}{u}-\frac{\mu-1}{r}\right)\frac{dv}{v^2}+t\left(\frac{1}{v}+\frac{\mu-1}{s}\right)\frac{du}{u^2}.\quad du+dv=0.$$

$$\therefore\left(\mu-\frac{t(\mu-1)}{r}\right)u^2+tu=\left(\mu+\frac{t(\mu-1)}{s}\right)v^2+tv.$$

The value of v from this equation in (1) gives the value of u.
If t, the thickness of the lense, be neglected, we get $u=\pm v$.
This in $\frac{1}{u}+\frac{1}{v}=(\mu-1)\left(\frac{1}{r}-\frac{1}{s}\right)=\frac{1}{f}$, gives $\frac{2}{u}=\frac{1}{f}$, or $u=2f=v$.
Taking the formula $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, $u+v=$minimum, we get

$$\frac{du}{u^2}+\frac{dv}{v^2}=0.\quad du+dv=0.$$

$\therefore u=\pm v$ and $u=2f=v$, as before. The object and image are both twice the focal distance from the lense.

Also solved by C. N. Schmall.

288. **Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.**

Find $\int\frac{xdx}{(1+x^3)^{2/3}}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; FRANCIS RUST, M S., Pittsburg, Pa.; and V.M. SPUNAR, Pittsburg, Pa.

Let $1+x^3=x^3z^3$. Then $\frac{xdx}{(1+x^3)^{2/3}}=-\int\frac{dz}{z^3-1}=u$.

Hence, $u=-\frac{1}{3}\int\frac{dz}{z-1}+\frac{1}{3}\int\frac{(z+2)dz}{z^2+z+1}=\frac{1}{6}\log\left[\frac{z^2+z+1}{(z-1)^2}\right]+\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2z+1}{\sqrt{3}}\right]$

$$=\frac{1}{6}\log\left[\frac{(1+x^3)^{2/3}+x(1+x^3)^{1/3}+x^2}{(1+x^3)^{2/3}-2x(1+x^3)^{1/3}+x^2}\right]$$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2(1+x^3)^{1/3}+x}{x\sqrt{3}}\right].$$

Also solved by J. Scheffer, S. G. Barton, and C. N. Schmall.
S. Lefsehetz and V. M. Spunar should have been credited for solving 284 and 285 in last issue.

MECHANICS.

237. **Proposed by C. N. SCHMALL, 604 East 5th Street, New York.**

In a naval action an officer observes that in the case of two guns firing, at elevations α and β, respectively, the projectiles of the former fall a feet short of the target while those of the latter land b feet beyond. The initial velocity being the same in both cases, prove that the *true* elevation is

$$\tfrac{1}{2}\sin^{-1}\left[\frac{a\sin 2\beta+b\sin 2\alpha}{a+b}\right].$$

(Suggested by problem 29, page 219, Jeans' *Theoretical Mechanics.*)

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

Let R=horizontal distance to target, V=initial velocity, g=the gravitational constant, and ϕ=elevation of gun to make the range equal to R.

Then the standard formula for the horizontal range gives for the elevations of ϕ, α, and β:

$$\begin{aligned}
&(1)\quad V\sin 2\phi=gR,\\
&(2)\quad V\sin 2\alpha=g(R-a),\\
&(3)\quad V\sin 2\beta=g(R+b).
\end{aligned}$$

Multiplying (3) by a, (2) by b, and adding, we obtain

$$V(a\sin 2\beta+b\sin 2\alpha)=gR(a+b)=V\sin 2\phi(a+b),$$

whence $\phi=\tfrac{1}{2}\sin^{-1}\left[\dfrac{a\sin 2\beta+b\sin 2\alpha}{a+b}\right]$.

Also solved by **H. C. Feemster, J. Scheffer, G. B. M. Zeer, S. G. Barton, and J. E. Sanders.**

238. **Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.**

Find the position of the center of pressure of a semi-elliptical area completely immersed in water, the bounding major-axis being inclined to the horizon at an angle β, and having one extremity in the surface of the water.

Solution by the PROPOSER.

If the area is not in the same vertical plane as the major axis, suppose it is inclined at an angle α. The ellipse projects into an ellipse in a vertical plane having the semi-axes a and $b\cos\alpha$.

Then from 229, pp, 189-190, Vol. XVI, No. 11, we get

$$=\frac{a(16b\cos\alpha\cos\beta+15a\pi\sin\beta)}{4(4b\cos\alpha\cos\beta+3\pi a\sin\beta)},$$

$$\bar{y}=\frac{b\cos\alpha(16\,a\sin\beta+3\,\pi\,b\cos\alpha\cos\beta)}{4(4b\cos\alpha\cos\beta+3\,\pi\,a\sin\beta}$$

The distance of the center of pressure below the surface of the water is

$$\frac{b\cos\alpha\cos\beta(16a\sin\beta+3\,\pi\,b\cos\alpha\cos\beta)+a\sin\beta(16b\cos\alpha\cos\beta+15a\,\pi\sin\beta)}{4(4b\cos\alpha\cos\beta+3\,\pi\,a\sin\beta)}$$

$$=\frac{32ab\sin\beta\cos\beta\cos\alpha+3\,\pi\,b^2\cos^2\alpha\cos^2\beta+15\,\pi\,a^2\sin^2\beta}{4(4b\cos\alpha\cos\beta+3\,\pi\,a\sin\beta)}.$$

Also solved by S. G. Barton.

239. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

A uniform bar of length $2a$ is placed in a sloping position, its lower end on the ground (coefficient of friction being μ), its upper end in the air, the bar being supported by a rough fixed peg (coefficient of friction μ'), against which it rests. If h is the height of the peg from the ground, and if θ be the angle the bar makes with the horizon, when on the point of slipping, prove that θ is to be found from the equation

$$\sin\theta\cos\theta\,[(\mu-\mu')\cos\theta+\sin\theta\,(1+\mu\mu')]=\mu\,h/a.$$

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Let R and R' be the pressures on the ground and peg, respectively, and W the weight. Then resolving the forces vertically and horizontally, and taking moments about the point on the ground, we have the three equations:

(1) $W=R+R')(\cos\theta+\mu'\sin\theta)$,

(2) $\mu\,R+R'(\mu'\cos\theta-\sin\theta)=0$,

(3) $R'h\csc\theta=Wa\cos\theta$, or $W=\dfrac{R'h}{a\sin\theta\cos\theta}$.

Substitute this value of W in (1), multiply the equation by μ and subtract (3) to eliminate R', and we have

$$\mu\cos\theta+\mu\,\mu'\sin\theta-\frac{\mu\,h}{a\sin\theta\cos\theta}-\mu'\cos\theta+\sin\theta=0.$$

Whence, $\sin\theta\cos\theta[(\mu-\mu')\cos\theta+\sin\theta(1+\mu\,\mu')]=\mu(h/a)$.

Also solved by G. B. M. Zerr and J. Scheffer.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

164. Proposed by G. J. GRIFFITHS, M. A., in Educational Times (Unsolved).

Prove that the sum of the squares of the reciprocals of all integers which are not divisible by the square of any prime is $15/\pi^2$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and V. M. SPUNAR, Pittsburg, Pa.

A solution of this problem is found on page 211, Vol. XIII, No. 11. Also page 134, Vol. V, No. 5.

The required sum is

$$S=\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\ldots$$

$$=\frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots}$$

$$1/\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\ldots=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots=\frac{\pi^4}{90}=\sigma_1.$$

$$1/\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\ldots=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}=\sigma.$$

$$S=\frac{\sigma}{\sigma_1}=\frac{\pi^2}{6}/\frac{\pi^4}{90}=\frac{15}{\pi^2}.$$

Also solved by S. Lefsehetz.

165. Proposed by J. EDWARD SANDERS, Weather Bureau, Chicago, Ill.

Factor (if possible), 11, 111, 111, 111.

Solution by the PROPOSER.

The prime factors of 11, 111, 111, 111 are 21649 and 513239. The best method known to me to find them, is to make an extended table of numbers of the form $22n+1$ and after striking out the composite ones use the remaining numbers as trial factors. The table may be extended as required but in any case need not be carried beyond 25000.

PROBLEMS FOR SOLUTION.

ALGEBRA.

335. Proposed by L. E. DICKSON, Ph. D., The Uuiversity of Chicago.

A person has \$1800 in notes payable \$18 monthly, bearing 10% interest. Find their present value if the interest is payable at the maturity of each note; also present value if interest is payable annually. [An actual business transaction.]

336. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Evaluate the determinant

$$\triangle=\begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \ldots & a_n^2 \\ a_2^2, & a_3^2, & a_4^2, & \ldots & a_{n+1}^2 \\ a_3^2, & a_4^2, & a_5^2, & \ldots & a_{n+2}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n^2, & a_{n+1}^2, & a_{n+2}^2, & \ldots & a_{2n}^2 \end{vmatrix}$$

GEOMETRY.

365. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Given the coordinates of the four vertices of the tetrahedron, (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3); (x_4, y_4, z_4): find volume and express it by a determinant.

366. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Astronomy, Manchester, N. H.

Construct a triangle, having given the base, vertical angle, and difference of altitude and difference of other two sides.

367. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

The tangents from a point A to a circle are bisected by a line XYZ, which cuts a chord in X and the tangents at its extremities in Y, Z. Show that $XAY=XAZ$, or $XAY=\pi-XAZ$. Also, reciprocate with respect to A.

CALCULUS.

292. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the partial differential equation, $x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}=axy$.

293. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find the length of the integral curve of the differential equation $(y^2x^{\frac{1}{3}}+2)dx-x^{\frac{1}{3}}\,dy=0$ between $x_1=1$ and $x_2=8$.

294. Proposed by C. N. SCHMALL, New York City.

Examine the function, $f(x)=\dfrac{(x-1)(x-2)}{(x-3)}$ and determine why its *minimum* value is *greater* than its maximum.

MECHANICS.

246. Proposed by A. M. HARDING, Adjunct Professor, University of Arkansas, Fayetteville, Ark.

A pentagon $ABCDE$, formed of equal uniform heavy rods connected by smooth joints at their ends, is supported symmetrically in a vertical plane with A uppermost, and AB and AE in contact with two smooth pegs in the same horizontal line. Prove that if the pentagon is regular, the pegs must divide AB and AE each in the ratio $1+\sin(\pi/10):3\sin(\pi/10)$. Jeans' *Theoretical Mechanics*, page 112, number 13.

247. Proposed by C. N. SCHMALL, New York City.

A cylinder of height h and radius r is standing on a horizontal seat in a railway car while the train is getting under way with an acceleration f. Show that the cylinder will not be disturbed if

$$f<\mu g, \text{ and } f<2rg/h,$$

where μ is the coefficient of friction.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

171. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve completely:

$$\begin{aligned} 2x^2-1&=y,\\ 2y^2-1&=z,\\ 2z^2-1&=w,\\ 2w^2-1&=x. \end{aligned}$$

172. Proposed by H. C. FEEMSTER, York, Neb.

Show that $\dfrac{(nr)!}{n!(r!)^n}$ is an integer.

173. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find integral values satisfying the equation,

$$a_1^2+a_2^2+a_3^2+\ldots+a_n^2=d^4.$$

NOTES AND NEWS.

Professor L. E. Dickson, who has been abroad during the past six months, returns to residence at the University of Chicago for the Spring Quarter, beginning April first, 1910. S.

The twenty-sixth regular meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, April 8, 9, 1910, at the Ryerson Physical Laboratory. Arrangements will be made for members present to dine together on Friday evening. S.

Professor H. B. Newson, of the University of Kansas, died suddenly at his home on February 18, at the age of fifty years. Professor Newson had held a full professorship since 1905 and would have succeeded to the headship of the department at the close of the present academic year. He was a graduate of Ohio Wesleyan University and had studied at Johns Hopkins, Heidelberg, and Leipsig. S.

At the annual meeting of the American Federation of teachers of the mathematical and physical sciences in Boston, during the Christmas recess, several important committees made preliminary reports, including the committee on a syllabus in geometry, the committee on college entrance requirements, and various committees under the International Commission on the teaching of mathematics. Two new committees were appointed, one to cooperate with the College Entrance examination board on the best form of logarithmic tables to be used in examinations, and one to consider the question of the publication of a journal devoted exclusively to secondary mathematics. Professor C. R. Mann, of the University of Chicago, was elected president for the ensuing year, and Professor E. R. Smith, of the Brooklyn Polytechnic Institute, secretary. S.

The series of articles which has been running for the past fourteen months in the *New York Independent*, on Great American Universities, was brought to a close in the March issue with a general article making some comparisons of great interest in respect to age, attendance, alumni, libraries, incomes, doctorates, etc. The author of the articles is Edwin E. Slosson, Ph. D. (Chicago). The *Independent* announces a similar series of articles in the near future on Great Foreign Universities. S.

W. B. Fite, assistant professor of mathematics in Cornell University, and H. E. Hawkes, assistant professor of mathematics in Yale University, were appointed full professors of mathematics in Columbia University. Professor C. J. Keyser has been given charge of the mathematical department of Columbia University, succeeding Dean Van Amringe, who retires from active service. M.

Professor Jules Molk, 8 Rue d' Alliance, Nancy, France, invites all mathematicians to mail to him corrections and additions to the published numbers of the great *Encyclopédie des Sciences Mathématiques*. Each volume is to contain "complements" on all the articles of the volume. It is also expected to include a comparative list of the principal technical terms used in the volume, in English, French, German, and Italian. The entire encyclopedia will probably fill more than thirty large volumes. Over 600 pages of the first volume and some parts of the following volumes have appeared. The work is published jointly by Gauthier-Villars of Paris, and B. G. Teubner of Leipzig. M.

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THE TEACHING OF THE CALCULUS.

By W. B. FORD, University of Michigan.

Of the various branches of collegiate mathematics doubtless none have received so much attention on the pedagogical side during recent years as the Calculus. In the newer text books on the subject, for example, unusual care has been taken to produce a high order of pedagogical as well as scientific excellence. The same theme has also played an important rôle of late years in the activities of scientific societies, an instance of which occurs in the title of a recent presidential address read before the American Mathematical Society, which was "The Teaching of the Calculus."*

In view of this wide-spread interest in the subject of the present paper, it is with no little hesitation that the author, in accepting an invitation from the Editors of the MONTHLY to express his own views, undertakes to add anything new or of value to it. However, there are two ideas or views which appear to him deserving of somewhat more prominence than they have yet received, and he gladly takes this occasion to at least give expression to them.

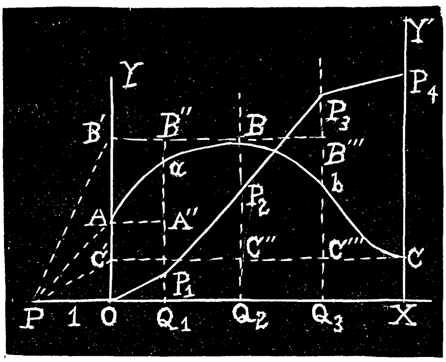
Referring again to the newer texts, we may say that they are distinguished from the old mainly in two particulars. First and foremost, they enliven the calculus by bringing it into closer touch with the physical, this in turn being accomplished through the systematic introduction of problems based upon every day experience or taken from the laboratory. Secondly, the logical flaws, chiefly those relating to infinitesimals which were so prevalent in the texts of a generation ago, have been eliminated through the introduction of closer cut definitions of the terms employed. Needless to say, the effect of these betterments has been widely felt and is constantly making its impress in our colleges today. However, there is doubtless room for still further improvement. The present situation may perhaps be described by saying that those pupils, relatively few in number, who really have the power to think may indeed understand the calculus now if they choose, but

*Address of April 27, 1907, by Professor Wm. F. Osgood. See *Bulletin American Mathematical Society*, Vol. 13, pages 449-467.

that the pupil who has only a fair measure of natural talent still finds it essentially difficult, though somewhat less abtract in its problems than formerly. This may necessarily be so on account of the subtleties of the subject, but I am inclined to the belief that without seriously impairing the dignity or the value of the calculus, it may and should be brought still closer to the comprehension of the average pupil.

In considering how this may be done, I am led to the first idea, mentioned above, which in substance is that the ordinary first course in calculus should be more largely graphical, not in the mere sense of employing frequent geometrical figures to illustrate analytical facts, but in the adoption of some of the ideas belonging properly to the so-called "Graphical Calculus." This latter subject, it may be said, is generally but little known to the pure mathematician, though its importance is of no mean order since it is fast becoming the recognized calculus of the engineer and the physicist. Though of comparatively recent origin, its growth has been very rapid and its methods, whatever else may be said of them, are such as to create a calculus which appeals easily to the average pupil and at the same time has a wide field of usefulness in technical work. It is not my intention to enter into a description of the subject here, since there are abundant sources of information concerning it. However, for the sake of definiteness, I will illustrate what is meant by graphical integration. The particular method followed is one recently brought out by Professor Carl Runge of the University of Goettingen during a course of lectures on graphical methods delivered recently at the University of Michigan.

In the following diagram let it be desired to find the area (given graphically) bounded by the curve ABC, the horizontal line OX and the vertical lines OY, XY.



We first lay off a unit length in the position OP. The point B is then determined by the eye as the one at which the curve has its maximum ordinate. We then determine, again by the eye, the points a, b so that the *area* $AA''a$=*area* $aB''B$. Then the required area=area under the rectilinear figure $AA''B''B'''\ C'''\ C$. To find this area graphically, project the points B, C on the line OY, thus obtaining, respectively B', C'. Connect P with $AB'C'$. Then from O draw the line OP_1 parallel to PA, cutting $A''B''$ in P_1. From P_1 draw the line P_1P_2 parallel to PB' cutting BC'' in P_2 and $B'''C'''$ in P_3. Finally from P_3 draw P_3P_4 parallel to PC' cutting XY' in P_4. The line XP_4 represents graphically the desired area.

The correctness of these results is readily established. Thus, in the

similar triangles POA, OQ_1P_1 we have $PO:OA::OQ_1:Q_1P_1$. But $PO=1$ and hence $Q_1P_1=$the area of the rectangle $OQ_1A''A$. Similarly, it may be shown that $Q_3P_3=$the sum of the areas of the rectangles OQ, $A''A$ and $Q_1Q_3B'''B''$, while $XP_4=$the latter sum plus the area of the rectangle Q_3XCC'''.

Greater accuracy may frequently be obtained by dividing each of the arcs AB, BC into several portions (instead of two) such that any one divides the area of its circumscribing rectangle into two equal portions. Again, the selection of a relatively large number of such points and the completion of the subsequent constructions evidently leads to a curve analagous to $OP_1P_2P_3P_4$ but representing a closer approximation to the true integral curve. Thus, the integral curve itself may be sketched in with considerable accuracy, thereby affording a means of computing instantly the area under the curve between any two given ordinates.

We shall not dwell upon the generality of this method of integration or upon various other features of it which are revealed by a close examination. It is to be remarked, however, that many important two dimensional determinations such as areas, centers of gravity, and moments of inertia, are admirably carried out by similar methods. Increased familiarity with the graphical calculus, in fact, always convinces one of its merits, and, as is natural, there are some who would even abandon quite completely the standard form of calculus in favor of the graphical. While it will be seen from what I have to say later on, that such an extreme in no wise seems desirable to me, it may be instructive to note the attitude of the graphical enthusiasts as illustrated in the following quotation taken from the introduction of a recent work.*

"All up to date teachers of engineering and applied science generally now recognize the vast superiority of the graphical over the purely mathematical methods of instruction of almost every description. . . . The attempt to employ purely mathematical, in preference to graphical, methods seems to me quite as absurd as attempting to teach geography by giving the position of towns in terms of their latitude and longitude and explaining the shape of a country by giving the equation of its coast-line instead of by employing the graphical method, that is, exhibiting a map."

The opinions here expressed certainly furnish food for thought. It is undeniably true that the curves with which one deals to-day in applied science are rarely if ever given by means of their equations, but rather graphically. They are determined experimentally by ascertaining a sequence of points satisfying prescribed conditions, and then drawing a smooth curve through them. It is the inability of the ordinary calculus to actually furnish the area under a curve when given in this manner, or to determine centers of gravity and moments of inertia for figures thus bounded—in short, to really *do* things—that is its fatal weakness for the engineer and physicist.

***Graphical Calculus*, by A. H. Barker. London, Longmans, 1905.

Enough has now been said to justify the following inquiry. In view of the recognized importance of the so-called graphical calculus in applied science and in view also of the fact that a large and increasing number of pupils who study the calculus to-day have technical careers in view, would it not be well for the calculus in its present scholastic form to recognize in larger measure the merits of the graphical method and incorporate some of its ideas? My own belief, as already indicated, is that such departures would be advisable. This does not imply an undervaluation of the logical aspect of the calculus but rather a proper valuation of the newer methods and processes which have come to play so important a rôle in it. Neither does it imply a revolution in our methods of instruction, but merely a new point of view in which present conditions, especially as regards the needs of the majority of our pupils, are more squarely met. But to what extent, it may be asked, do you really mean to change the ordinary first course? Would you omit some of the present material in order to make room for the graphical? Doubtless, all would be willing to introduce some of the latter *if there time for it.* Of this I shall have more to say in my second thought, mentioned at the beginning and which I now proceed to consider.

All mathematicians recognize in their scientific work the value of the process of successive approximation. I would now ask whether this process should not be better recognized also on the pedagogical side. In the teaching of the calculus, for example, can we after all expect the average pupil in his first course to appreciate the modern point of view as regards the infinitesimal? True it is that clearness cannot be assured to the scientific mind without such refinements, but should we expect to make the Calculus clear in this sense to a beginner? However heterodox I may appear, I do not believe so. Such insight, in fact, belongs usually to later stages in one's development. The question which the beginner desires answered is, what is the calculus about *about*— what does it *do?* The attitude is the same for any branch of scientific study when first approached and it is only after such questions have been answered that there is a natural inquiry into or need for the really *exact.* Not until then is it time for the further approximations to that end. Moreover, many who are studying the Calculus to-day — in fact the large majority — never reach this second stage at all for lack of natural endowment, yet they have a perfect right to study the subject and are capable of deriving great profit from it. The distinctly higher approximations, leading ultimately into the modern theory of functions of a real variable, are, of course. for those only who possess primarily theoretical inclinations. I believe, therefore, that a more thoughtful application of the successive approximation principle to our teaching would improve it. Likewise, our texts might perhaps be bettered in this particular. They, likewise, should be written in the knowledge that the average beginner is concerned with facts, not with reasons why. Thus it may be said that while our newer texts have indeed brought the calculus nearer home to

the average pupil through the introduction of problems based upon daily experience or upon the experiments of the laboratory, still more will stand to the credit of our time when a different arrangement of material has been produced — one, in fact, which will better recognize the successive approximation principle on the pedagogical side. To be more specific, let us be satisfied in the beginning with a rough definition of the derivative, allowing intuition and not rigor to be the chief factor in supplying it and therefore drawing very largely upon the geometrical side. Included in this first approximation should be some simple problems in maxima and minima and the first notions of the integral calculus, all following the same general plan of putting emphasis upon the "what" and *not* the "why." This may be done within the first few weeks of the course after which the second approximation is to be entered upon. In this the whole field should be enlarged and strengthened logically as far as time will permit. It is in this latter connection that opportunity should be found to introduce something at least of the graphical methods as outlined above. Depending upon whether the class is primarily of technical students or otherwise, the teacher should use his judgment as to where the chief emphasis in this second approximation should fall. Conceivably, there are classes in which one might eventually develop the notion of rigor, even in a first course, to such an extent as to acquaint the student with the meaning of an arithmetic *epsilon* proof, while there are classes in which the graphical should largely predominate at the expense of the analytical. Neither element can be omitted, but more judgment should be used in proportioning them.

In recognizing, therefore, the great advance which has been made in later years toward a correct teaching of the Calculus, we would merely point out two features in which still further improvement seems possible. However suicidal it may seem to the analyst, the first course in calculus should be framed to meet more adequately the mental endowments of the average pupil. In particular, we should give a larger recognition to the merits of the so-called graphical calculus. We should also recognize more fully that it is the facts of the calculus and not the logical coherence of it that appeal primarily to the beginner, and this in turn necessitates a fuller recognition on the pedagogical side of the principle of successive approximation.

A NEW THEOREM IN THE GEOMETRY OF CONICS.*

By ALAN SPENCER HAWKESWORTH.

If two or more triangles, to which any conic is in common escribed, have each a vertex upon one and the same axis of said conic, and if, in each case, a circle be described, passing through the other two vertices of the said triangle, and with its center upon the said axis:

Then will the resultant system of circles have the other axis of the conic for their common radical axis, and will pass in common through the two foci, real or imaginary, as the case may be, lying upon that axis, while their "limiting points" upon the first and given axis, and line of centers of the said system of coaxal circles, will be, in turn, the imaginary or real foci of the conic.

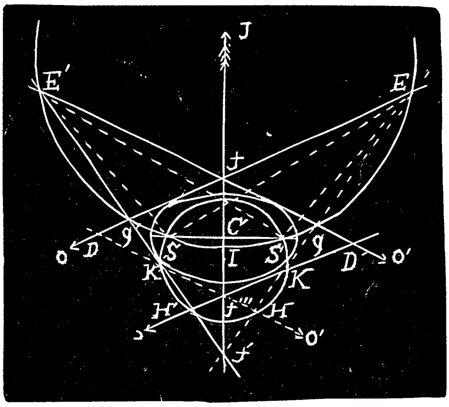


Fig. 1.

(A) For, first, take any central conic (Fig. 1), and let Et, Et' be any two tangents to it, concurring in E, and cutting its minor axis in points t and t', respectively. Draw $E't$ and $E't'$, two other tangents, symmetric about the minor axis with Et and Et', and thus concurring with them in t, t', respectively upon the axis. And let Egt' and $E'tg$ cut in g; and $E'g't'$ and Etg' cut in g', forming a pair of congruent triangles Etg and $E'tg'$, to which the conic is escribed; with a common apex t upon that conic's minor axis. Join S and S' the conic's two foci, to points E and E', in lines SE, SE', $S'E$ and $S'E'$.

Then, by a well known theorem, and by symmetry,

$$\text{angles } SEg = S'E'g' = S'Eg' = SE'g.$$

So that points $EgSS'g'E'$ are concyclic about a center lying on tt' their axis of symmetry, the conic's minor axis.

And in like manner, for any other two pairs of tangents, symmetric about that minor axis. Say, for example, $t'HK$ and $t'H'K'$; $Ht''K'$ and $H't''K$; concurring upon the minor axis in $t't''$, and cutting in KK'. Whence as before,

$$\text{angles } SHK = S'H'K' = S'HK' = SH'K,$$

and thus $HKSS'K'H'$ are concyclic about a center lying on the minor axis.

*Read before the American Mathematical Society, September, 1909.

So that, conversely, if from any point t upon the minor axis of a conic, two tangents Etg' and $E'tg$ be drawn, and these be cut in E, g, respectively, by any third tangent to the curve, forming thus a triangle, with a vertex t upon the minor axis, to which the said conic is escribed, then the circle, whose center is on the minor axis, and which passes through the other two vertices E, g of said triangle, will also pass through the conic's two foci S, S'.

And hence, if two or more such triangles be drawn, as for example, Etg and $Kt''H$, then all such circles $EgSS'$ and $HKSS'$, passing in common through the two foci, have the conic's major axis for their common radical axis. While C, the conic's center, will be the position both of its imaginary foci, and of the now imaginary common "limiting points" of the said system of circles.

(B) Secondly, taking now any conic, ellipse, hyperbola, or parabola (Fig. 2), again let us have any two tangents ET and ET', concurring in E, but now cutting the conic's major axis in points T and T', respectively. Once more let us draw two fresh tangents $E'T$ and $E'T'$, but symmetric now about the major axis to ET and ET', and thus concurring with them upon it in T and T', respectively. And let them cut ETg' and $E'g'T'$ in g', and EgT' and $E'Tg$ in g; thus forming a symmetric pair of congruent triangles ETg and $E'Tg'$, to which the conic is escribed, each having a common vertex T upon that conic's major axis.

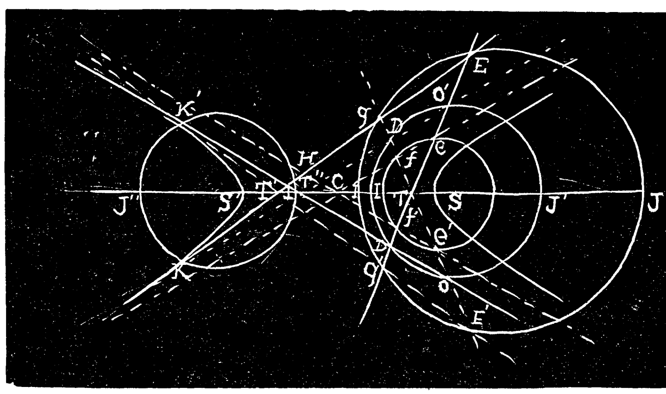


Fig. 2.

Let I and J be the two common excenters of the said two congruent triangles upon the conic's major axis, their axis of symmetry. Then IgJ, $Ig'J$, IEJ, and $IE'J$ being right angles, points I, g, E, J, E', g' are concyclic about the medial point of IJ upon the said major axis. And if the conic's foci S, S' be joined to, say E, in lines SE, $S'E'$, then, as before, $SEg'=S'Eg$; so that $SEI=S'EI$, and $SI:S'I=SE:S'E=SJ:S'J$ (Euclid VI, 3 and A).[1]

Conversely, therefore, the circle through Eg, whose center is on TT', the conic's major axis, ever harmonically divides the inter-focal distance SS' in points I and J. Or $CI:CS=CS:CJ$, whatever may be the triangle ETg, to which the conic is escribed, and which has its vertex T upon that conic's major axis.

Thus, for example, if $KT'H'$ (Fig. 2) be such a triangle, and the circle through KH', whose center is on the major axis, cuts that axis in points I', J', then, as before, $CI':CS'=CS':CJ'$.

From this it follows that the given conic's minor axis is the common radical axis of all such circles, and their common "limiting points" are the two foci of the conic. While the imaginary foci, lying at that conic's center, are now also the imaginary common intersections of that minor axis by any and all the circles.

Corollary 1. Hence when the conic is a circle, CS being now infinitesimal, all such circles must pass through its center. This fact is also evident from the equality of the angles CET and CEg; so that C is ever the common excenter of all the triangles escribed to the circle which has it for its center.

Corollary 2. On the other hand, with a parabola, any and all such circles must be concentric about the focus S, a fact which can also be shown as follows: The external angle ETg of the isosceles triangle ETE' is double either of the two equal interior angles TEE' or $TE'E$. And hence the center of the circle through $EE'g'g$ must lie concyclic to the points E, T, g (Euclid III, 20), while, by symmetry, it must also lie on axis TS. And next, by a well known theorem, the circumcircle of the circumscribing triangle ETg must pass through S, the focus of the parabola. From which it follows that the center of the circle through $EE'g'g$ must be either T or S. Now it cannot be T, since Tg and TE being equally inclined to the axis, the tangent EgT', by the nature of the parabola, must be so inclined that it cuts TE at a greater distance from that axis and from T than it cuts Tg. Hence S must be the center desired. And in like manner it can be shown that S is also the center of the circles through $KH'HK'$, and through $ODD'O'$, etc.

Corollary 3. So that were any circumscribing triangle DEH to an unknown conic given, and any transversal $tt't''$ or $TT'T''$ of that triangle posited as the position of an unspecified axis of unknown magnitude of the said unspecified conic, then there is ever an unique solution (with the exception yet to be noted in Corollary 4, following) of one, and one conic only, which shall be inscribed or escribed, as the case may be, to the given triangle, and have the given transversal for an axis, major, or minor, as the case may be.

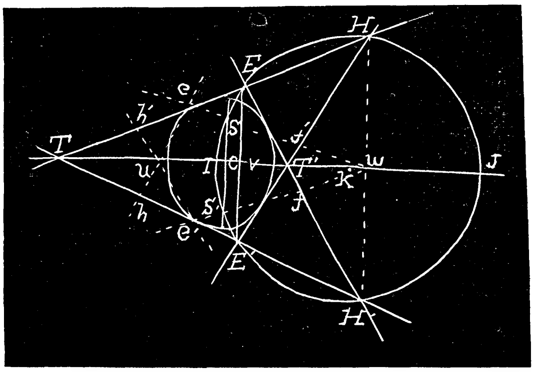
For describing a second circumscribing and congruent triangle $D'E'H'$, symmetric about the given axis, and cutting the first triangle in points t, t', t''; or T, T', T''; g, g'; K, K'; and O, O', we obtain three pairs of triangles ETg and $E'Tg'$, $HT'K$ and $H'T'K'$, DTO and $D'TO'$, each with a vertex upon the given axis. Wherefore, by the foregoing theorem, the circles through $Egg'E'$, $HKK'H'$, and $DOO'D'$ must determine the other axis; cutting, and passing in common through the foci, when the given axis happens to be the conic's minor, but none cutting, when the given axis happens to be a major, in which latter case the external radical axis, and its two real "limiting points" give, as stated, the minor axis and foci desired.

And the foci being thus found, a perpendicular from either upon any one of the six known tangents will give us, by an elementary theorem, a point on the conic's auxiliary circle, and hence the magnitude of the major axis. So that the desired conic can now be readily drawn.

Corollary 4. But an exception to the foregoing Corollary 3 obtains when the said transversal happens to pass through a vertex of the given circumscribing triangle.

For, if it pass through a vertex, without also bisecting the angle, then the corresponding two sides of the resultant symmetric triangle form in the said vertex four concurrent rays. And no conic is possible, other than in the degenerated form of a right line, coinciding with the given axis.

While if the given axis bisects, internally, or externally, an angle of the given circumscribing triangle, then the corresponding two sides of the symmetric triangle coalesce with the two sides of the given triangle, and a symmetric quadrilateral $HE'H'ET'T$ is formed (Fig. 3). An identical figure being given, whether the said axis bisects internally the common vertical angle HTH' of the two symmetric and congruent triangles ETH and $E'TH'$, or bisects externally in common the vertical angles $E'T'H$ and $ET'H'$ of the two symmetric and congruent triangles $E'T'H$ and $ET'H'$.



Hence the conic is as yet entirely indeterminate, without further data. For we now have but one pair of symmetric and congruent triangles $E'T'H$ and $ET'H'$, having a common vertex T' upon the given axis, and thus but one circle through the points E', H, J, H', E, I, upon which the desired conic's foci, real, or imaginary, must lie. And innumerable conics are thus possible, having the given axis for either a major or a minor.

But if, in addition, we are given the position of C, the center of the required conic; then, if C falls within the limits I, J, the two excenters upon TT' of $E'T'H$ and $ET'H'$, a perpendicular through it will cut the circle through $IE'HJH'E$ in S, S', the desired foci, and TT' will thus be the minor axis of the desired unique conic.

While, with C falling anywhere outside IJ along TT', then the perpendicular through it will now be the minor axis of the required unique conic, whose major is TT'. And the fixed ratio $CI\cdot CJ = CS^2 = CS'^2$ gives us the desired foci.

Furthermore, we can point out the limits of C, with regard to any desired conic, as follows.

Join $E'E$ and HH', cutting, and being perpendicularly bisected by TT' in points v and w, respectively. Let u be the medial point of magnitude TT', and let het, $h'e't'$ be the symmetric and congruent medial triangles of HET and $H'E'T'$, respectively.

Then, obviously, he and $h'e'$ bisect in common TT' in u; and, in like manner, hvt and $h'vt'$ concur in v, and etw and $e't'w$ concur in w, and not otherwise. Were the symmetric medial triangles of $E'T'H$ and $ET'H'$ taken, their produced sides must likewise concur in the said three points u, v, w, along TT'.

Lastly, let K be the center of the circle through points I, E', H, J, H', E. We will show this to be also the common intersection of TT' by the circumcircles of ETH, $E'TH'$, $E'T'H$, and $ET'H'$.

(a) Then, with the given center C at either I or J, there evidently results either the common incircle or excircle of ETH and $E'TH'$, or one of the two common excircles of $E'T'H$ and $ET'H'$, as the case may be.

(b) With C at either v or w we have, in each case, a degenerated right line conic, having foci at E', E or H, H', as the case may be; each of which may be considered as the extreme form of, or limit between, an ellipse and a hyperbola.

(c) The center C falling anywhere between I and v gives us, as shown, foci falling symmetrically upon the arc $E'IE$, so that we now have an ellipse, inscribed to ETH and $E'TH'$, or escribed to $E'T'H$ and $ET'H'$, with TT' as its minor axis; while, in like manner, C falling anywhere between w and J gives an ellipse, with TT' as its minor axis, and foci which fall symmetrically upon arc HJH', escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$.

(d) And similarly, C falling anywhere between v, w gives us a hyperbola, whose minor axis is TT', escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having foci upon the symmetric arcs $E'H$ and EH', and the asymptotal angles which rise from zero, as C moves from v to T', attain the maximum value of $E'T'H$ when C is at T', where $ET'H$ and $E'T'H$ are the asymptotes; and sink again towards zero as C moves from T' to w; since, by the law of the hyperbola, real tangents concurring upon the minor axis, and thus touching opposite branches of the curve, must ever concur in an angle less than the supplementary asymptotal angle (this being their ideal maximum), and hence must concur supplementarily in an angle ever greater than the said asymptotal angle.

(e) Next, taking C at u, the medial point of TT', we plainly obtain, once more, a degenerated ellipse or hyperbola, a right line conic, but now one coincident with TT', its major axis, and having foci at T, T'; a fact also evident from the consideration that $IT':IT=JT':JT$, or $uI\cdot uJ=uT^2=uT'^2$.

(f) From this it follows that if C be taken anywhere within the limits u and I, then the foci S, S' must fall, one between T, I and the other between T', I, to fulfill our ratio $CI\cdot CJ=CS^2=CS'^2$, so that we now obtain an ellipse, inscribed to ETH and $E'TH'$, or escribed to $E'T'H$ and $ET'H$, and having TT' for its major axis.

(g) While, in like manner, C falling anywhere along TT', produced beyond J, gives us an ellipse, escribed in common to ETH and $E'TH'$, or to

$E'T'H$ and $ET'H'$, having TT' for its major axis, and foci which fall between J, K [as we will later prove], and upon TT' produced beyond J respectively.

(h) On the other hand, if C be taken along TT' produced, anywhere beyond u, the same ratio $CI \cdot CJ = CS^2 = CS'^2$ necessitates that its two foci S, S' must fall outside the limits T, T', along TT' produced in both directions, so that now we obtain a hyperbola, escribed in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having TT' for its major axis; and an asymptotal angle, which rises from zero, as C moves from u towards T, attains its maximum value of HTH' when C is at T, and HT, $H'T$ are the asymptotes; and sinks again towards zero as C retreats from T along TT' produced towards infinity. For obviously, as before stated, by the law of the hyperbola, the concurrent angle HTH' of the tangents TH and TH', which touch the same branch of the curve, must ever be greater than their ideal asymptotal angle.

(i) Lastly, escribing a parabola in common to ETH and $E'TH'$, or to $E'T'H$ and $ET'H'$, having TT' for its major axis, then, in either case, its focus S, by a well known theorem, must lie on the circumcircles of the said circumscribing triangles ETH and $E'TH'$, or $E'T'H$ and $ET'H'$; while by the foregoing Corollary 2, this focus S must also be the center of the circle through $E'HEH'$, and hence K, this center, is also the common intersection of axis TT', as stated, by the four circumcircles of ETH, $E'TH'$, $E'T'H$ and $ET'H'$. And thus, when the center of our conic has retreated to infinity in either direction along TT', then K will be the position of the focus of the resultant parabola, escribed to ETH and $E'TH'$, or to $E'TH$ and $ET'H'$.

(h) So that, returning to the previously mentioned hyperbola, whose center is beyond u, along TT' produced, and whose major axis is TT', it further follows that one of its foci must ever fall between the limits T', K, and thus within the circumcircles of its circumscribing triangles ETH and $E'TH'$, or $E'T'H$ and $ET'H'$; and similarly, as stated, one focus of the ellipse, escribed to the same triangles, must fall within the limits K, J and outside those circumcircles. Results independently proved in a previous theorem of mine (*Supplemento ai Rendiconti del Circolo Matematico di Palermo*).

Corollary 5. Wherefore, were we given any two tangents to an unspecified conic, Et and Et', or ET and ET', concurring in E, and were also given the position of the said conic's two axes, and thus its center, but knew neither the magnitude, nor the names of the said axes, then in every case (with the exceptions yet to be noted, in Corollary 6) a unique solution is given by the foregoing theorem, that is, one conic and one only, fulfilling the required conditions of touching the two given tangents, and having its axes in the given positions.

For, letting Et and Et' (Fig. 1) or ET and ET' (Fig. 2) be our two given tangents, with tCt' and ACA', or TCT' and BCB', as the two unspecified axes, we can, obviously, take either of these two axes as the axis of symmetry for our second pair of tangents $E't$ and $E't'$, or $E'T$ and $E'T'$, cutting

our first pair in t, g and g', t', or in T, g and g', T', respectively; and giving us then two congruent and symmetric triangles Etg and $E'tg'$, or ETg and $E'Tg'$, whose vertical angles at t or T are externally bisected by the said axis of symmetry tt' or TT'. From which it follows, as shown, that the circle through $EgIg'E'J$ either cuts the other axis ACA' in S, S', the foci of the required conic; or else, if it cuts that other axis merely in imaginary points, then the ratio $CI \cdot CJ = CS^2 = CS'^2$ gives us these foci.

And note: if the said circle through $EgIg'E'J$ has its center on the conic's major axis, then C is external to IJ; while if its center is on the minor, then C is within IJ; and hence, were we to draw EI, or EJ, the internal or external bisector of the concurrent angle tEt' or TET', according to whether C is nearer to I or to J, then the said bisector EI or EJ will ever cut and cross the major axis of our conic before it can cut the minor in I or J, as the case may be.

So that, conversely, with the given concurrent tangents Et and Et', or ET and ET', and the two unspecified axes, if the concurrent angle tEt' or TET' be bisected, internally, or externally, by EI or EJ, as the case may be, then this bisector will ever cut and cross first that axis which is to be the major of our unknown conic, an important fact to know.

But, if the said bisector should happen to pass through C, and cut neither axis first, then the conic must thereby be the incircle or excircle of tEt' or TET', according to whether the said bisector is the internal or external bisector of tEt' or TET'.

Lastly: with C as a known point, the desired conic is plainly central. But were one of the two axes and C specified as lying at infinity, then we must evidently have a parabola, and the medial point K of the points I and J wherein our internal and external bisectors of TET' cut our known axis TT', will be its focus, and the parabola can thus be drawn.

Oorollary 6. But, as in Corollary 4, an exception to the foregoing evidently occurs in the special case of the concurrent point of our given two tangents falling upon one of our given axes.

For if the said axis does not bisect the concurrent angle, then we merely obtain, by axial symmetry, two sets of concurrent rays, and no conic is possible, other than in the degenerated form of the said axis itself.

While, with the axis bisecting internally or exteraally the concurrent angle, then the twofold axial symmetry gives us a rhombus* within which we can plainly place an incircle and innumerable inscribed ellipses, each having either of the two axes for its major, while either axis can also be the major axis of innumerable escribed hyperbolas, whose asymptotal angles vary from zero, up to an ideal maximum, that they can never reach, equal to that of the rhombic angles which lie on the chosen major axis.

Corollary 7. In the theorem, the circle whose diameter is the "limit-

*So that our conic is here indeterminate. And further data are needed.

ing point," and thus the inter-focal distance of the real or imaginary foci, as the case may be, plainly cuts all the co-axial circles orthogonally.

Corollary 8. In the theorem, the hyperbola's asymptotes being also tangents, the circle through the points (say e, f, e', f') in which any two tangents cut them, is also one of the co-axial system having the minor axis for its radical axis. So that $Ce.Cf = CS^2 = Ce'.Cf'$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

NOTE ON SHORT METHODS IN ARITHMETICAL CALCULATIONS.

From time to time, we have sent to us for publication, "short cuts" and "lightning methods." Most of these are of no theoretical and little practical value, and hence cannot be given a place in the MONTHLY.

The short cuts below were sent to us by Mr. Charles H. Case of Chicago, who is now neariy 81 years old. He says the list, consisting of eighteen examples, was prepared for the students of Wheaton College, November, 1896. We publish a few of them because we have found some of them useful in practical computation, having used them for years.

Mr. Case says, "The examples given should be wrought without the use of more figures than are used in the same." The principles used may be found mainly in the algebraic formulae given below.

$$(a \div b)(b \div a) = 1; \quad (a+b)^2 = a^2 + 2ab + b^2; \quad (a+b)(a-b) = a^2 - b^2).$$

1. $(5\frac{1}{2})^2 = 30\frac{1}{4}$; $(7\frac{1}{2})^2 = 56\frac{1}{4}$; $(65)^2 = 4225$; $(88)^2 = 7744$; $96^2 = 9216$; $36^2 = 1296$; $76^2 = 5776$.

2. $625^2 = \begin{array}{l} 360625 \\ \underline{\;3\qquad\;} \\ 390625 \end{array}$; $876^2 = 767376$; $2496^2 = 6230016$.

3. $$\begin{array}{r} 99649964125^2 = 9928129699281296015625 \\ +198562592 \\ +249124910 \\ \hline 9930115350113787015625 \end{array}$$

4. $(2436)^2 = \begin{array}{l} 5761296 \\ \underline{\;1728\quad\;} \\ 5934096 \end{array}$; $68 \times 132 = 8976$; $8852 \times 8948 = 79207696$; $868 \times 932 = 808976$.

5. $48\frac{17}{29}\times 49\frac{12}{29}=2400\frac{697}{841}$; $1295\frac{57}{93}\times 1296\frac{36}{93}=1677615\frac{7343}{8649}$.

6. $7464\times 7536=56248704$; $88044\times 87956=7744000000-1936$.

7. $2777\frac{7}{9}\times 41666\frac{2}{3}\times 666\frac{2}{3}\times 54\times 24\times 52\times 7692307\frac{9}{13}\times 625\times 125\times 56\times 32\times 142857142857142 8\frac{4}{7}\times 2083\frac{1}{3}\times 48\times 833\frac{1}{3}\times 3125\times 68543764287590=\ldots$"

These and eleven other even longer computations are carried out mentally by Mr. Case.

The principle, $a^2=(a-b)(a+b)+b^2$, may be used in the squaring of any number, though it is not so readily used if the numbers consist of more than two digits. Thus,

$$87^2=(87-3)(87+3)+3^2=84\times 90+9,$$
$$92^2=(92-2)(92+2)+2^2=90\times 94+4.$$

This is the principle used in several of Mr. Case's calculutions. Thus, $(5\frac{1}{2})^2=(5\frac{1}{2}-\frac{1}{2})(5\frac{1}{2}+\frac{1}{2})+(\frac{1}{2})^2=30\frac{1}{4}$. ED. F.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n\doteq\infty} x^{1/(2n-1)},\ n=1,\,2,\,\ldots;\quad f(x)=\lim_{n\doteq\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

Remark by the PROPOSER.

Professor F. H. Safford, of the University of Pennsylvania, has sent me the following expressions for the function defined in the problem:

$$\frac{2}{\pi}\int_0^\infty \frac{\sin xz}{z}dz,\quad \frac{2}{\pi}\int_0^\infty \frac{xdz}{x^2+z^2},\quad \lim_{m=+\infty}\frac{e^{xm}-e^{-xm}}{e^{xm}+e^{-xm}}.$$

333. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the infinite series

$$\frac{1}{(m+1)^2}+\frac{(2m-1)}{(2m+1)^2}+\frac{(3m-1)^2}{(3m+1)^4}+\frac{(4m-1)^3}{(4m+1)^5}+\frac{(5m-1)^4}{(5m+1)^6}+\ldots$$

[No solution of this problem has been received.]

334. Proposed by G. B. M. ZERR. A. M., Ph. D., Philadelphia, Pa.

Sum the series, $2^n-n\,2^{n-2}+\frac{n(n-3)}{2!}2^{n-4}-\frac{n(n-4)(n-5)}{3!}2^{n-6}$

$$+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-8}-\frac{n(n-6)(n-7)(n-8)(n-9)}{5!}2^{n-10}+\ldots$$

Solution by the PROPOSER.

We have, $(1-px)(1-qx)\equiv 1-x(p+q-pqx)$. Taking logarithms,

$$\log(1-px)+\log(1-qx)=\log[1-x(p+q-pqx)].$$

Hence, $(p+q)x+\dfrac{(p^2+q^2)x^2}{2}+\dfrac{(p^3+q^3)x^3}{3}+\ldots+\dfrac{(p^n+q^n)x^n}{n}$

$$=x(p+q-pqx)+\frac{x^2(p+q-pqx)^2}{2}+\ldots+\frac{x^n(p+q-pqx)^n}{n}.$$

The coefficients of x^n are equal for values of x, which makes the series convergent.

$$\therefore \frac{p^n+q^n}{n}=\frac{(p+q)^n}{n}-\frac{(n-1)pq(p+q)^{n-2}}{n-1}$$

$$+\frac{(n-2)(n-3)p^2q^2(p+q)^{n-4}}{2!(n-2)}-\frac{(n-3)(n-4)(n-5)p^3q^3(p+q)^{n-6}}{3!(n-3)}+\ldots$$

Hence, $p^n+q^n=(p+q)^n-npq(p+q)^{n-2}$

$$+\frac{n(n-3)}{2!}p^2q^2(p+q)^{n-4}-\frac{n(n-4)(n-5)}{3!}p^3q^3(p+q)^{n-6}+\ldots$$

Let $p=q$. Then $2=2^n-n2^{n-2}+\dfrac{n(n-3)}{2!}2^{n-4}-\dfrac{n(n-4)(n-5)}{3!}2^{n-6}$

$$+\frac{n(n-5)(n-6)(n-7)}{4!}2^{n-8}-\ldots$$

Solved similarly by V. M. Spunar, who starts with the identity, $2\log(1-x)=\log(1-2x+x^2)$.

GEOMETRY.

359. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two tangents are drawn to two confocal parabolas from any point on a common tangent. Show that the former two tangents and their chord of contact envelop yet another confocal parabola.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let S be the focus, T the point on the common tangent; TP, TQ the tangents to one parabola; TR, TU the tangents to the other parabola; TP, $1R$ being the common tangent. Then S, P, Q, R, U, T are concyclic.

Hence, since this circle passes through S, T, Q, U, it passes through the intersections T, Q, U of the three lines TQ, TU, QU.

$\therefore$ The parabola that is tangent to TQ, TU, QU has S for its focus. (Todhunter, *Conic Sections*, Art. 146.)

II. Solution by S. LEFSEHETZ, East Pittsburg, Pa.

First solution [Fig. 1]. Let F be the common focus, T_1 and D_1, T_2 and D_2 the tangents and directrices of the given parabola; M a point on the common tangent; MC_1 and MC_2 the two tangents drawn by ordinary construction to the two parabolas; C_1 and C_2 their contacts; C_1a_1 and C_2a_2 the perpendiculars drawn from them to the respective directrices; P_1 and P_2 their intersection with T_1 and T_2. By well known properties of parabolas we know that FP_1 is perpendicular to MP_1.

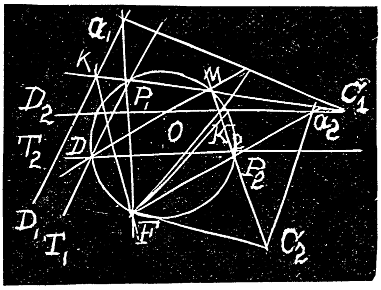


Fig. 1.

$\therefore \angle FP_1T_1 = \angle a_1C_1P_1 = \angle P_1C_1F$.

Also, $\angle FP_2T_2 = \angle P_2C_2F$. Now MC, the common tangent, passes through the point where T_1 and T_2 meet, since FD perpendicular to MC must meet it on both these lines. Therefore in circle O (drawn on MF as diameter) $\angle DP_1F = \angle DP_2F = \angle P_1C_1F = \angle P_2C_2F$.

$\therefore \angle MC_1C_2 = \angle FC_1C_2 = \angle K_1FK_2$ since K_1F and K_2F are respectively perpendicular to FC_1 and FC_2.

$\therefore$ Circles circumscribed to triangles K_1MK_2 and C_1MC_2 meet both in F, and if we apply Simpson's theorem, we find that the perpendiculars drawn from F to MC_1, MC_2, K_1K_2, C_1C_2 have their feet on a straight line. Therefore F is the focus of a parabola touching these four lines—one more than required.

Second solution. Transform by reciprocal polars, taking F for center of transformation (Fig. 2). Then our conics become circles O and O_1 meeting in F and in another point A. We have now to prove that MN being a

common chord of circles O and O_1, passing through A, MC and NC, respective tangents in M and N, B point where OM and O_1N meet, the five points F, B, M, C, N, are on the same circle. That B, M, N. C are on the same circle is evident. Then $\angle AFM = \angle CMA$, $\angle AFN = \angle ANC$.

$\therefore \angle MFN = \angle CMA + \angle CNA = 180° - \angle C$, which proves that F is on circle $BMNC$, and therefore the proposition.

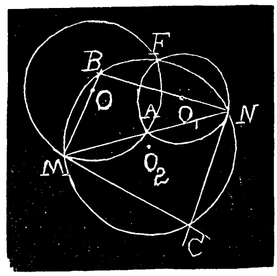


Fig. 2.

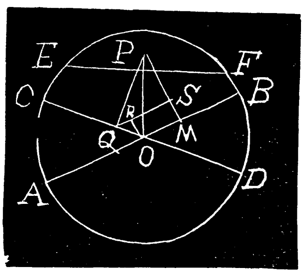
This is in all its generality, for we can readily see that B is the transformed of K_1K_2—line joining points where the tangents meet the respective directrices.

360. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A circular segment, area A, revolves successively about the diameters (fixed) d, d', intersecting at an angle θ. If v =volume about d, v' the volums about d', then $v^2+v'^2-2vv'\cos\theta$ is independent of the position of the segment.

Solution by S. LEFSEHETZ, East Pittsburg, Pa., and the PROPOSER.

Let P be the center of gravity of the segment; EF, its chord; O the center of the circle; and AB, CD, the diameters d, d', respectively; $\angle AOC = \angle BOD = \theta$.



Draw PQ perpendicular to CD, PM perpendicular to AB, QS perpendicular to PM, and OR perpendicular to QS.

Let $PQ=a$, $PO=c$, $PM=b$.

Then $b=PS+SM=PS+OR=a\cos\theta+\sqrt{(c^2-a^2)}\sin\theta$. $v=2\pi Ab$, $v'=2\pi Aa$.

$\therefore v^2+v'^2-2vv'\cos\theta = 4\pi^2A^2(a^2+b^2-2ab\cos\theta) = \triangle$.

$$\therefore \triangle = 4\pi^2A^2[a^2+a^2\cos^2\theta+2a\sin\theta\cos\theta\sqrt{(c^2-a^2)}+(c^2-a^2)\sin^2\theta-2a^2\cos^2\theta-2a\sin\theta\cos\theta\sqrt{(c^2-a^2)}].$$

Hence, $\triangle = 4\pi^2A^2c^2\sin^2\theta$.

Solved similarly by S. G. Barton, J. Scheffer, and A. H. Holmes.

CALCULUS.

Remark on 282, by F. H. SAFFORD, Ph. D., University of Pennsylvania.

The published solution of 282 is incomplete. It is valid for a *long* box, but with a *short* box, the outer corner, B, may not reach its maximum before the corner, A, in Dr. Zerr's figure *emerges* from the hall. See Fig. 2, page 186, October, 1907. In that figure, N is omitted but should be vertically *above* Q. S is the corner vertically *under* M. In line 2, page 187, omit b preceding the word, becomes.

289. Proposed by G. W. DROKE, Professor of Mathematics, University of Arkansas.

Find the curve such that the rectangle under the perpendiculars from two fixed points on the normals be constant.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

This is problem 27, page 61, Murray's *Differential Equations.*

Take the straight line through the fixed points as the x axis, and the middle point as the origin. The coordinates of the points are then $(a, 0)$ and $(-a, 0)$. Using p for $\frac{dy}{dx}$, the equation of a normal at (x', y') is

$$y-y'=-\frac{1}{p}(x-x').$$

Its distance from $(a, 0)$ is

$$\frac{\frac{a}{p}-\left(y'+\frac{x'}{p}\right)}{\sqrt{1+\frac{1}{p^2}}},$$

and its distance from $(-a, 0)$ is

$$\frac{\frac{-a}{p}-\left(y'+\frac{x'}{p}\right)}{\sqrt{1+\frac{1}{p^2}}}.$$

The constant product, dropping primes, is

$$\frac{y^2+\frac{2xy}{p}+\frac{x^2}{p^2}-\frac{a^2}{p^2}}{1+\frac{1}{p^2}}=k^2,$$

or $p^2y^2+2xyp+x^2-a^2=k^2p^2+k^2$, or $(y^2-k^2)p^2+2xyp+x^2-a^2-k^2=0$;

whence the p discriminant is

$$4x^2y^2-4(y^2-k^2)(x^2-a^2-k^2).$$

Hence, $\quad 4x^2y^2-4x^2y^2+4(k^2+a^2)y^2-k^2(x^2-a^2-k^2)=0,$

$$\text{or, } \frac{y^2}{k^2}+\frac{x^2}{k^2+a^2}=1.$$

This is the singular solution of the differential equation. It represents a system of confocal conics having the fixed points as foci.

Also solved by V. M. Spunar, G. B. M. Zerr, and J. Scheffer.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

However the ellipse is placed relative to the co-ordinate axes, its area is

$$S=\int_{x_1}^{x_2}(y''-y')\,dx,$$

where x_1 and x_2 $[x_1<x_2]$ are the extreme abscissas taken in the curve, and y' and y'' $[y'\leq y'']$ are the two ordinates corresponding to any one value of x. Solved for y, the given equation becomes

$$by=-(hx+f)\pm\sqrt{[f^2-bc+2(hf-bg)x+(h^2-ab)x^2]}.$$

Now let

$$f^2-bc=b^2A,\quad fh-bg=b^2B,\quad ab-h^2=b^2C,$$

where for an ellipse $C>0$. Then the difference of the two ordinates becomes

$$y''-y'=2\sqrt{[A+2Bx-Cx^2]}.$$

Hence, integrating,

$$S=2\left[\frac{Cx-B}{2C}\sqrt{[A+2Bx-Cx^2]}+\frac{B^2+AC}{2C^{\frac{3}{2}}}\cdot\sin^{-1}\frac{Cx-B}{\sqrt{[B^2+AC]}}\right]_{x_1}^{x_2}.$$

Now the extreme abscissas make $y'=y''$, or $A+2Bx-Cx^2=0$; whence

$$Cx_2=B+\sqrt{[B^2+AC]}\ \text{ and }\ Cx_1=B-\sqrt{[B^2+AC]}.$$

Substituting these values,

$$S=\frac{B^2+AC}{C^{\frac{3}{2}}}[\sin^{-1}1-\sin^{-1}(-1)]=\frac{\pi}{b}\,\frac{[(fh-bg)^2+(ab-h^2)(f^2-bc)]}{(ab-h^2)^{\frac{3}{2}}},$$

which immediately reduces to the formula proposed.

Also solved by G. B. M. Zerr and J. Scheffer.

MECHANICS.

240. Proposed by S. A. COREY, Hiteman, Iowa.

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of

that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

[No solution of this problem has been received.]

240. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at both ends, is loaded in the form of a parabola, height of vertex b. Find deflection at center due to this load.

Solution by the PROPOSER.

Let AB be the beam, ACD the parabola, $CD=b$, $AD=DB=a$. Take E any point on AB, draw EF perpendicular to AB. Let $AE=x$, and also z be the distance of the center of gravity of the area AEF from EF. Then $\frac{4}{3}ab$=total load. $(x-a)^2+(a^2/b)(y-b)=0$ is the equation to the parabola, with A as origin.

$$\text{Then } (x-z)=\frac{\int\int xdxdy}{\int\int dxdy}=\frac{\int\int xdxdy}{A}.$$

$$\therefore A(x-z)=\int_0^x xydx=\frac{b}{a^2}\int_0^x(2ax^2-x^3)dx=\frac{bx^3}{12a^2}(8a-3x).$$

$$A=\int_0^x ydx=\frac{bx^2}{3a^2}(3a-x).$$

$$\therefore x-z=\frac{8ax-3x^2}{4(3a-x)} \text{ and } z=\frac{4ax-x^2}{4(3a-x)}.$$

Taking moments about A we get,

$$EI\frac{d^2y}{dx^2}=\tfrac{2}{3}abx-Az=\tfrac{2}{3}abx-\frac{bx^3}{3a}+\frac{bx^4}{12a^2}=M.$$

$$EI\frac{dy}{dx}=\tfrac{1}{3}abx^2-\frac{bx^4}{12a}+\frac{bx^5}{60a^2}+C.$$

When $x=a$, $dy/dx=0$, $C=-\frac{4}{15}a^3b$.

$$\therefore EI\frac{dy}{dx}=\tfrac{1}{3}abx^2-\frac{bx^4}{12a}+\frac{bx^5}{60a^2}-\tfrac{4}{15}a^3b,$$

$$EIy=\tfrac{1}{9}abx^3-\frac{bx^5}{60a}+\frac{bx^6}{360a^2}-\tfrac{4}{15}a^3bx=-\frac{61a^4b}{360} \text{ when } x=a.$$

$$\therefore y = -\frac{61a^4b}{360EI} = \text{deflection required.}$$

For cantilever beam, length $2a$, with the same load,

$$EI\frac{d^2y}{dx^2} = -A(x-z) = -\frac{2bx^3}{3a} + \frac{bx^4}{4a^2},$$

$$EI\frac{dy}{dx} = -\frac{bx^4}{6a} + \frac{bx^5}{20a^2} + C; \text{ when } x=2a,\ dy/dx=0,\ C=\tfrac{16}{15}a^3b.$$

$$\therefore EI\frac{dy}{dx} = -\frac{bx^4}{6a} + \frac{bx^5}{20a^2} + \tfrac{16}{15}a^3b.$$

$$EIy = -\frac{bx^5}{30a} + \frac{bx^6}{120a^2} + \tfrac{16}{15}a^3bx = \tfrac{24}{15}a^4b, \text{ when } x=2a.$$

$$\therefore y = \frac{24a^4b}{15\,EI} = \text{deflection at end of beam.}$$

Also solved by Harold Rowe.

242. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

In a certain New York theatre there is an asbestos curtain supported by thin circular rings, radius r, which move on a cylindrical rod of radius a. The curtain is intended to be drawn by a *steady pull.* Taking μ as the coefficient of friction, show that this will not be possible if r be less than $a\sqrt{(1+\mu^2)}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let P be the resultant pull on a ring, θ the angle between the direction of P and the normal to the surface of contact of ring and rod.

Then $P\cos\theta$=resolved part of P along the normal, and $P\sin\theta$=resolved part at right angles to the normal. For equilibrium, $P\sin\theta < \mu P\cos\theta$.

$$\therefore \sin^2\theta < \mu^2\cos^2\theta, \text{ or } \cos^2\theta > \frac{1}{1+\mu^2}.$$

The diameter of the ring is in the direction of P; diameter of rod is the normal.

$$\therefore \cos\theta = \frac{a}{r}. \quad \therefore \frac{a^2}{r^2} > \frac{1}{1+\mu^2} \text{ or } r < a\sqrt{(1+\mu^2)}.$$

243. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A weight W is supported by three strings of the same size and quality lying in the same plane. The middle string is vertical, one string makes with it an angle θ on one side, and the other string makes with it an angle ϕ on the other side. Find the stresses T_1, T_2, T_3 in the strings.

244. Proposed by C. N. SCHMALL, New York City,

In a game of billiards a player observes two balls, A and B, at rest in a certain position and concludes that it would be to his advantage to project A against B in such a manner that, as a result of the impact, A might suffer the greatest possible deviation from its course. Taking the balls to be equal and smooth, each of diameter a and elasticity e, and the distance between their centers to be d, show that he can accomplish the desired motion by projecting A in a line making an angle equal to the

$$\sin^{-1}\frac{a}{d}\sqrt{\frac{1-e}{3-e}}$$

with the line joining the centers.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology.

Let α be the angle between the direction of the motion of the impinging ball and the line joining the centers at the time of impact, and x the required angle. We then have, from the law of sines,

$$\frac{\sin x}{\sin\alpha}=\frac{a}{d}, \text{ or } x=\sin^{-1}\left[\frac{a}{d}\sin\alpha\right].$$

When a moving smooth ball strikes another smooth ball of the same weight at rest, we have that the tangent of the angle between its original and new directions is (*c. f.* Bowser's *Analytic Mechanics*, problem 30, page 387),

$$\frac{1+e}{2}\cdot\frac{\sin 2\alpha}{1+\sin^2\alpha-e\cos^2\alpha}.$$

The deviation will be a maximum when its tangent is a maximum. Neglecting the constant, and differentiating for the maximum, we find the condition to be

$$[1+\sin^2\alpha-e\cos^2\alpha]2\cos 2\alpha-\sin^2 2\alpha[1+e]=0,$$
$$\text{or, } 2[(1-e)+(1+e)\sin^2\alpha]\ [1-2\sin^2\alpha]-4\sin^2\alpha[1-\sin^2\alpha]\ [1+e]=0,$$
$$\text{or, } 2[1-e]-2[3-e]\sin^2\alpha=0;$$

whence for the maximum deviation,

$$\sin\alpha=\sqrt{\frac{1-e}{3-e}}, \text{ and } x=\sin^{-1}\frac{a}{d}\sqrt{\frac{1-e}{3-e}}.$$

Also solved by G. B. M. Zerr.

NOTES AND NEWS.

The fourth article in the series on the teaching of collegiate mathematics is found in this number, namely: The Teaching of Calculus, by Professor W. B. Ford, of the University of Michigan. Intimately related to this article is a review of an English text on the Calculus by Dr. N. J. Lennes of the Massachusetts Institute of Technology. The fifth article of the series comes next month by Professor Frederick H. Bailey of the Massachusetts Institute of Technology on Unified Mathematics.

Prof. Leonard E. Dickson, formerly editor of the MONTHLY, has just returned to residence at the University of Chicago after a leave of absence which he spent abroad. He has been elected Chairman of the Chicago section of the American Mathematical Society for the present year.

Dr. N. J. Lennes, of the Massachusetts Institute of Technology, has received an appointment as instructor in mathematics at Columbia University, beginning in the autumn of 1910. During the summer he will be in charge of the mathematical courses at the Chautauqua Summer School, Chautauqua, N. Y.

At the spring meeting of the Chicago section of the American Mathematical Society, held at the University of Chicago, April 8-9, 1910, seventeen papers were read and about fifty members were in attendance.

The demand for trained teachers of mathematics for colleges and universities seems far in excess of the supply. At least such is the report from the larger universities which turn out such teachers. The present year is no exception. Already the calls are numerous and urgent. The word would seem to be that more young men should take advantage of the opportunity and at once enter upon a course which will provide the adequate training to meet such demands.

Mr. Egbert J. Miles has been appointed to an instructorship in mathematics at Cornell University, Ithaca, N. Y.

Mr. Arthur D. Pilcher, who has been on leave of absence from the University of Kansas during the past two years, will return to his work in the department of mathematics there with the opening of the autumn semester, 1910.

Professor Archibold Henderson, of the University of North Carolina, has been granted a leave of absence for the year 1910-11.

Professor E. J. Wilczynski, of the University of Illinois, will give courses in mathematics at the University of Chicago during the summer quarter, 1910.

Dr. Anthony L. Underhill, of the University of Minnesota, has been appointed instructor for the summer quarter, 1910, at the University of Chicago.

Miss E. R. Bennett, Fellow at the University of Illinois, has been appointed instructor in mathematics at the University of Kansas.

At the summer session at the University of Illinois, the following courses in mathematics will be offered: Advanced Algebra, Plain Trigonometry; Teachers' Course for Secondary Teachers, Plain Analytic Geometry, Differential Calculus, Integral Calculus, Differential Equations, Elementary Theory of Groups, Theory of Equations and Determinants, Seminar and Thesis Course.

Can any of our subscribers furnish us a copy of No, 1, Vol. XI of MONTHLY. Also a copy each of Nos. 6 and 11 of Vol II ? We have calls for these numbers and we are willing to pay a fair price for them. F.

Professor Oscar Bolza has resigned at the University of Chicago and will return to Germany, where he will lecture in the University of Freiburg. He has been identified with the University of Chicago since its foundation in 1892, and his loss will be felt by a large number of students who love him for his kindly friendship and his inspiring teaching, as well as admire him for his distinguished scholarship in his chosen field of research. Included in this body of his students are no less than seven of his present colleagues at Chicago, who will feel his loss even more keenly than those who are situated in other parts of the country.

The Central Association of Science and Mathematics Teachers is emphasizing the concrete presentation in all branches, and in this spirit the mathematics section has been collecting, through a committee, an extended list of real applied problems in algebra and geometry, from a variety of sources. These have been published in *School Science and Mathematics* during the past two years and have now been gathered and classified in a small pamphlet and are being tried out by a large number of teachers. Copies of this pamphlet may be secured in any quantity, at five cents each, from the secretary of the Mathematical Section, Miss Mabel Sykes, 1223 East 57th street, Chicago, Ill.

Dr. George Bruce Halsted has the work of revising the definitions of the mathematical terms and the selection of new mathematical words for the two new volumes of the *Century Dictionary* which are now going through the press. Some of the new words introduced in current mathematical literature have been coined by Dr. Halsted. F.

"Periodic Orbits about an Oblate Spheroid" is the title of an extended article in the January, 1910, number of the *Transactions of the American Mathematical Society*, by Dr. W. D. MacMillan of the University of Chicago.

Mr. E. B. Escott, instructor in mathematics at the University of Michigan, has an article on "Logarithmic Series" in the *Quarterly Journal of Pure and Applied Mathematics,* No. 162, 1910, and also an article on "Cubic Congruences with Three Real Roots" in the *Annals of Mathematics* for January, 1910.

Dr. E. J. Wilczynski, of the University of Illinois, has been appointed to an associate professorship in mathematics at the University of Chicago. He will enter upon his work at the beginning of the Summer Quarter, 1910.

The Summer Quarter, 1910, at the University of Chicago, will begin Monday, June 20. The courses in mathematics include Trigonometry, College Algebra, Synopic Course in Pure and Applied Mathematics, Differential Calculus, Integral Calculus, Theory of Equations, Differential Equations, Critical Review of Secondary Mathematics, Graphical Methods in Algebra, Theory of Substitutions, General Analysis, Functions of a Complex Variable, Modern Analytic Geometry, Projective Differential Geometry, Seminar on the Foundations of Mathematics, and Reading and Research in Pure Mathematics. The first term extends to July 27, and the second term to September 2.

The courses in advanced mathematics at the various American and foreign universities are announced with some regularity in the *Bulletin of the American Mathematical Society.* Those for 1910—1911 at Cornell, Princeton, Yale, and the University of Strassburg are found in the May, 1910, issue. In this same number also is a list of the doctorates for the year 1908—1909 conferred by the various German universities.

Mr. A. S. Hawkesworth has been appointed professor of higher mathematics in the University of Pittsburgh, Pittsburgh, Pennsylvania.

This issue was mailed May 21.

ERRATA.

The statement of the following problem should appear at the top of page 95.

290. Proposed by C. N. SCHMALL, New York City.

When the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, represents an ellipse, show (by integration) that its area is

$$\frac{\pi\,(af^2+bg^2+ch^2-abc-2fgh)}{(ab-h^2)^{\frac{3}{2}}}.$$

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UNIFIED MATHEMATICS.

By F. H. BAILEY, Massachusetts Institute of Technology.

One of the first questions which a student asks at, or near, the beginning of his study of any branch of mathematics is, "What is it good for?" In most cases he is not satisfied unless he can be shown some specific utilitarian use to which his new knowledge may be put. With many students this attitude of mind persists throughout their study, which, for lack of a satisfying answer, is often closed at an earlier date than is desirable. Other students continue their mathematics solely for the use they can make of it as a tool in their study of some other science, as physics, for example.

Good pedagogy requires that the solution of concrete problems should form a large and important part of any course of mathematics, not merely because it will increase the student's interest in his work and at the same time give him the ability to use his mathematics as a tool, but because it will aid him in acquiring the power of close and accurate reasoning, of careful analysis, and of seeing the mathematical relationships of quantities — in a word — mathematical power. But with this end in view the mere results of the problems, which the utilitarian type of mind would prize so highly, become relatively insignificant. and the methods of attack and solution are of chief importance. In fact the extreme utilitarian, omitting everything which does not seem directly applicable to some "practical" problem, and in what he retains, focusing his attention upon the results, will defeat his own ends. For his students will be left with a few rules applicable in the particular cases studied, but they will have little or no mathematical ability— no power of analysis and no general theory for attacking new types of problems.

It must be acknowledged, however, that a student of good mathematical ability, or one who has been well taught and has followed a well arranged program of mathematical study, oftentimes is apparently unable to apply his mathematics to the study of a new science — in other words, he seems to lack that mathematical habit of mind at which we have aimed. In many cases, however, this failure is only apparent and the real difficulty is

with the method followed by the teacher of the new science. For example, we sometimes hear teachers of physics complain that their students cannot use mathematics in the solution of problems in physics. Before asking the students to make mathematical solutions of physical problems, the physicist must be sure that his students have been made familiar with the new concepts of physics, in their mathematical elements as well as their purely physical elements, and have seen that his teacher finds it of advantage to use mathematics in the solution of problems. In that way his own enthusiasm will be aroused to attempt the same methods, and he will not be called upon to apply his mathematics to data which are very dimly outlined in his mind. Indeed, one of the great difficulties in teaching mathematics is the selection of problems the concepts of which are thoroughly familiar to the student so that he can be asked to apply his mathematics intelligently. Yet the teacher of physics, sometimes forgetting how long it takes a student to become familiar with a new concept, even a very simple one, may override this fundamental principle of mathematical work—a clear knowledge of the data of a problem—and ask the student to apply his mathematics at too early a date. Not until he is sure that he has made his students appreciate the physical data has the teacher of physics any right to complain of their mathematical work.

While, then, the results of problems should not be regarded as of fundamental importance, the solution of problems, taken from all possible sources, must form a large part of any course in mathematics, since in this way the power of analysis is largely developed. This power of analysis—the recognition of the mathematical element in a problem—and a good understanding of the meaning of mathematical symbols and the laws governing their relations should be the two aims of a course in mathematics. Some students enjoy the problem work more than the theoretical work, and others care more for the theoretical work than the problem work, but both classes of students must study the subject on both its sides if they would not fall short of their possibilities, even in the particular part of the work in which they are the more interested.

In planning a course of mathematics, then, what order shall be used? In trying to answer this question we shall consider mathematics through a first course in calculus, *i. e.*, arithmetic, algebra, plane geometry, solid geometry, trigonometry, analytic geometry, differential calculus, and integral calculus. It is usual to teach these subjects separately and in the order in which they have been named, with the exception that the Algebra is often divided into two courses, the second course being placed after the course in plane geometry. This method divides this portion of the science up into distinct parts or units, and the student appreciates this division probably more strongly than he appreciates the unity of the whole science. How, otherwise, are we to explain the inability, or the unwillingness, of the average student to use the mathematics of one division in the work of an-

other division? For example, after a year's study of algebra, plane geometry is taken up. Many of the proofs in plane geometry may be expressed very neatly in algebraic form; also many propositions in geometry open up a field of problems for algebraic solution. Now it is well known that with many students it is most difficult, if not impossible, to make them use their algebra in this work. They seem to regard the algebra as if it were finished and laid aside, and not as a vital part of their mental training, to be used everywhere that it will be of advantage.

Recently, however, even in elementary algebra considerable use has been made of the graph in explaining results and in forecasting what may be expected as the answer to any problem. It may be noted in passing that, if the student is to derive the most benefit from this work, care should be taken to prevent him from regarding the graph as merely an interesting appendix at the end of the problem, but to make him think of it as an integral part of the solution. Here analytic geometry has aided algebra in the solution of its problems in a very valuable manner, and at the same time the barriers between these two branches of mathematics have been partially broken down, although the student, of course, does not appreciate this fact.

Again, when any single division of mathematics is developed by itself, such development is apt to be narrow and even to give a wrong impression. For example, analytic geometry may be studied without any use of the calculus, and a very compact and interesting body of methods and results may be put together. But the difficulty of dealing with any but the simplest curves before the methods of calculus are used has made analytic geometry and conic sections almost synonomous terms in the past. Moreover, the student's knowledge of the meaning of analytic geometry will in this way be defective, unless he goes on to the study of calculus, and he will be obliged throughout his life to use poorer tools than is necessary. For example, tangents and normals to curves are found by methods which are intrinsically those of calculus, but which can be applied only in the simplest cases, and even then with considerable superfluous labor, because of the lack of the notation of calculus. If it should be argued that the student would have two methods, *i. e.*, the method used in analytic geometry, and the method of calculus, the reply is the question: "Does any one, after he knows how to determine tangents and normals by calculus, ever resort to the longer and more laborious methods he used in analytic geometry?"

While the study of each part of mathematics by itself will give the student a good idea of the development of thought in that particular field — a point which may be regarded by some as of decided advantage — such classification of mathematics may be perfectly well postponed to a later date, when the student has a certain body of mathematics which he knows and wishes to classify. At this later date, if he is interested, the student may with considerable pleasure and profit study the historical development of each part of his subject, for he will then have the necessary mathematical

maturity, and having, moreover, knowledge of more than one field he can compare the advances in the various fields, and note how an advance in one field made possible an advance in another field.

Mathematics is a science, an old and fundamental science, which is being used more and more in the advancement of other sciences. But this increased use in other sciences can give us little aid in determining a method of teaching mathematics—in fact, a teacher keeping this view alone in mind might meet the same kind of failure that the mere utilitarian meets, and fail quite as completely to achieve his object. It would seem, then, as if we must study mathematics as a science in itself, but find some method of obviating difficulties such as have been noted above. In looking for a new method we will first note the attitude of the professional mathematician toward his own science. As he proceeds in the study of his subject he usually finds that some type of the work appeals to him especially and devotes the greater part, if not all, of his energy to that field. But this specialization would not be apt to take place till after the first course in calculus. Before that time, he may, to be sure, be more interested in the development of the theory than in its application to problems, or vice-versa. The main thing to be noticed, however, is that *he always tries to make his solution of any problem, theoretical or practical, as direct and simple as possible, and pays little or no attention to the particular field, or fields, from which he takes his method.* If, then, the mathematician does not permit himself to be restricted to any one field for his method, why should we teach mathematics in the old divisions, the boundary lines of which have proved difficult for so many students to cross? Why set up any boundary at all for the student? Teach him his mathemetics as a unit, the only way in which he will use it, and promise him at a later date to classify it as he wishes.

In co-operation with others the writer has recently had the opportunity of laying out and conducting such a unified course of mathematics for the first two and one-half years of an engineering school. The course takes the place of distinct courses which were formerly given in theory of eqations, plane and solid analytic geometry, differential calculus, integral calculus, and differential equations. The results with the classes have fully justified the experiment, and with each succeeding year the writer's belief in the desirability and the effectiveness of such courses has increased.

In arranging a unified course the order is *an order of difficulty in the problems,* and the theory is developed as rapidly as it is required for the solution of the problems. The power of analysis and the ability to do formal work can be as logically developed in one order as in another. In addition there is the decided advantage that the same method may occur several times with problems of increasing difficulty, and by this repetition a method, at first vague, may become clear and powerful. For example, the tangent to a curve is first found for a curve of the type

$$y=a_0x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n,$$

where the right hand member is a rational algebraic polynomial of degree n, simple numerical cases being taken first; then for a curve of which the equation is algebraic, but more complex; and finally for curves of which the equations are transcendental. The first case requires only the formula for differentiating the algebraic polynomial; before the second case can be studied, the differentiation of any algebraic function must be studied; and before the last case can be studied, the differentiation of the transcendental functions must be considered. Here the whole subject of differentiation has been developed at the same time that a use for it has been shown, and the problem of the tangent has been studied at three different times with problems arranged in an ascending order of difficulty. Of course other problems would be carried on in this same manner with the development of the formulas of differentiation.

In a unified course there may be a saving of time, as no problem will be taken up till the best method of work has been developed for that problem. But such economy of time ought not to be regarded as the chief end to be attained by such a course. If it is so regarded, the course may be open to the criticism that some parts of mathematics, well and thoroughly studied before, may be slighted in the new course. As a matter of fact everything in the old courses, if desired, may be retained in the new course, and receive quite as much attention as in the old order, though one cannot but feel that some things as ordinarily given in the old courses could be omitted with no serious loss.

Again, the unified course, if too rapid advance is attempted, may be thought to make too great demands upon a student's power of assimilating new material. But it is in this very respect that the unified course is notably superior to the older style of course. In the latter course the particular new point is dwelt upon at such length in the effort to make the student understand it thoroughly before going on, as he will not take it up again, that he may find his work monotonous, and lose his interest. In the unified course, however, he takes up the new idea, studies it in an elementary way, and learns as much of it as possible without having the work become dull and uninteresting. He then goes on to some new ideas and after a time comes back to the study of the first idea with fresh pleasure in recognizing an old friend in new surroundings and with an increased appreciation of its full content. The old style course, by its intensive method, makes as great (if not greater) demands upon a student's power of assimilation, for it requires him to learn the new idea at once, while the unified course permits him to become acquainted with a new idea gradually, by meeting it at several different times, as illustrated by the example of the tangent line; and each time something is added to his previous knowledge, which is thus given time for a natural growth.

One other criticism of the unified course occurs to the writer at this time — the difficulty of a student referring to a text arranged for a unified course, when at some later time he wishes to use some formula or rule which he has forgotten. This difficulty, however, can easily be met by the compiling of a good index, and every student ought to learn the use of such an index, instead of shuffling the pages of his book till he finds what he wants. If the student of a unified course wishes to refer to some of the special texts of algebra, geometry, calculus, etc., it might be possible that he would not know what texts to consult — not know whether he ought to look in an analytic geometry or a calculus. But this difficulty seems so unlikely that it is hardly worth considering.

It is not to be assumed that all mathematics through the first course in calculus should be arranged in a single unified course. In fact, the break in going from the secondary school to the college seems to be an argument against such a large consolidation and to point to the desirability of at least two such unified courses.

Finally, such unified courses in the mathematics through the first course in Calculus are directly in line with what has long been done in the higher mathematics, as shown by the courses in analysis given by the French, and to some extent by the Germans, for advanced students. Surely a method which is so valuable there ought to be considered for other grades of work, unless the teaching of elementary mathematics is regarded as perfect at the present time; and the lively discussion of questions of teaching by the various associations of teachers makes one believe that the teachers are not so content with the present results.

JORDANUS NEMORARIUS AND JOHN OF HALIFAX.

By L. C. KARPINSKI, Teachers College, Columbia University.

Cantor closes the first volume of his *Geschichte der Mathematik*[1] with the names of Leonard of Pisa and Jordanus Nemorarius, stating that these men mark a new period in the development of mathematical science. Cantor's discussion[2] of the part played by Jordanus in opening a new era rests largely on the Schonerus edition[3] of an anonymous mediaeval *Algorismus Demonstratus* which was long attributed to Regiomontanus, for no other reason apparently than that Regiomontanus made a copy of a Vienna manu-

1). Vol. I, 3d edition, p. 911.

2). Vol. II, 2d edition.

3). D. E. Smith, *Rara Arithmetica*, pp. 178-179. Curtze (note, p. 20, Einige Materialen zur Geschichte der Mathematischen Facultaet der alten Universitaet Bologna, S. Gherardi, translation by Max. Curtze, Berlin, (1871), states that there is a XIV cent. Ms. in the library of Basle.

script of the same, which copy was used by Schoener for his edition[1]. The text is now attributed to an unknown Magister Gernardus[2]. Enestroem states[3] that the *Algorismus demonstratus* seems to have been unknown to the older writers on the history of mathematics. In a later article[4] he points out that the question as to which Algorismus Jordanus wrote seems to be very complicated, since there are several extant which are attributed to him[5].

In looking over the work of Wallis[6] on the early history of our numerals, chapter IV, *Quam antiquus sit harum in his regionibus Figurarum Numeralium usus,* I chanced upon a reference to an *Algorismus demonstratus* by Jordanus which is of sufficient importance to warrant giving the extract in full[7]. It should be noted that Wallis was a scholarly writer upon whose information we are able to rely. This reference antedates the work of Chasles,[8] mentioned by Enestroem as the first notice, by over one hundred and fifty years. The quotation is taken from the English edition of the Algebra which appeared in 1685.

"We have also, in Manuscript," says Wallis, "another treatise of *Algorism,* of *Jordanus,* (whom Vossius placeth about the year 1200 and contemporary with that *Campanus,* who wrote *De Computo Ecclesiastico* ;) entitled *Algorismus Jordani, tam in Integris quam in Fractionibus, demonstratus* ;[9] in which, the use of these Figures, and the way of numbering by them, is with great accuracy described and demonstrated. Which *Algorism* of his is very different from his *Arithmetica,* published and illustrated by *Faber Stapulensis* ; yet so, as it may very well be judged, by his manner of demonstration, to be a work of the same man. And the Manuscript itself, as appears by the hand and by the shape of the Figures, is very ancient. And in the same Manuscript book, wherein that of *Jordanus,* and some other small pieces are written, I found at the end of it two Celestial Schemes, relating to the year 1216 ; the one of them is called *Figura Anni,* representing the position of the Heavens on *March* 22, 1216 ; the other, *Figura Conjunctionis Saturni et Martis,* shewing the position of the Heavens at the time of that Conjunction which happened the same year, *October* 4, 1216. They are both of them described by these Numerical

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1). G. Enestroem, *Ist Jordanus Nemorarius Verfasser der Schrift "Algorithmus Demonstratus?"* Bibl. Math. 5_3, pp. 9-14.

2). P. Duhem, Sur l'Algorithmus Demonstratus, Bibl. Math. 6_3, p. 13.

3). L. c., note p. 9.

4). *Uber die "Demonstratio Jordani de Algorismo."* Bibl. Math. 7_3, pp. 24-30.

5). See also G. Enestroem, *Uber zwei angebliche mathematische Schulen im christlichen Mittel alter,* Bibl. Math. 7, pp. 252-262, and *Uber eine dem Jordanus Nemorarius zugeschrievene kurze Algorismusschrift,* Bibl. Math. 8_3, pp. 135-153. P. Treutlein, Abhandlungen zur Geschichte d. Math. Vol. II.

6). Johannis Wallis, Opera Mathem., Vol. II, Oxford, 1693.

7). In a personal letter from Mr. Enestroem to the author he states that this book is still in existence, Codex Savil. 21, Oxford, and that it was examined by Professor Bjornbo in 1906. It contains the text of the *Demonstratio Jordani de Algorismo.*

8). Apercu historique etc., Brussels, 1837.

9). The Latin version, Opera Omnia, vol. II, p. 13, is *Algorismus Jordani, tam in integris quam in fractis, demonstratus.* It adds also *Codex Saviliensis.*

Figures; and, in likelihood, were calculated about that time, in order to some Astrological Predictions to be made thereupon. And it so happens, that this last page of that Piece, proves to be the latter leaf of that same piece of Parchment, which begins that Book of *Algorismus Demonstratus* and therefore later written than it."

This statement by Wallis further complicates the matter under discussion,[1] as in the other works ascribed to Jordanus there is no special mention in the title of fractions, and in the treatment of fractions the name is given either as minutiæ philosophicæ or phisicæ (Sexagesimal fractions) and minutiæ vulgares (common fractions).

At the same time it would seem to show that some Jordanus did write an Algorismus about the beginning of the thirteenth century. Of men by the name of Jordanus several of about this time are well known in Church History[2] and, doubtless, there were others outside of the church. Jordanus Nemorarius is identified by Cantor with Jordanus de Saxonia (c. 1190-1237) of the order of Dominicans, based partly upon some early chronicles and partly upon the fact that this work often appears in the same volume with other works which are quite certainly the works of the Dominican. The name Nemorarius nowhere appears in the letters or writings of Jordanus of Saxonia, and the derivation of the appellation Nemorarius from his birthplace[3] rests upon an error in assuming the same to be Borrenstrick.

If the Saxon Jordanus did write any of these algorisms it is highly probable that this work had considerable influence in spreading the knowledge of our numerals throughout Europe, for as head of the order of Dominicans this man made many journeys over all of Europe; indeed he perished in a shipwreck in February, 1237, on the return voyage from a trip to the Holy Lands.[4] He had a reputation as a brilliant orator and is said to have made a thousand converts to his order. It would indeed be rather surprising if, in his wanderings, he had not somewhere learned and discussed the new symbols which were then coming into use.

Concerning the influence and even the identity of Jordanus we are much in doubt. In regard to the influence of John of Halifax there is no doubt. Maximilian Curtze,[5] who made a special trip through European libraries in search of Mathematical MSS., found that there are forty-five copies of the Algorismum of Sacrobosco in the three libraries of Munich, Vienna, and Erfurt (Amploniana). The Coxe catalog of an Oxford library gives ten further and there are doubtless numerous others extant in Europe. Of those mentioned at least seven are of the thirteenth century and the

1). See, however, note 7 above.

2). Potthast, *Bibliotheca Historica Mediaevi*, mentions four who lived between 1175-1300.

3). By Cantor, vol. II, J. J. Bertheir, *B. Jordanis de Saxonia, Opera, Friburg*, 1891, states that J. was born in Westphalia, not Borrenstrick. L'abbe E. Bernard, *Les Dominicains dans l'Universite de Paris*, Paris, 1883, largely on Jordanus.

4). Acta Santorum, Feb. 2, vol. II, pp. 720 ff.

5). *Petri Philomeni de Dacia in Algorismum vulgaren Johannis de Sacrobasco Commentarius*, Copenhagen, 1897, Max. Curtze.

others of later date up to the sixteenth century. In America there are two copies, one in the Plimpton Collection[1] and the other in the Columbia University Library. Both of these copies are the work of the fifteenth century.

In addition to the Algorismus of John of Halifax there are numerous commentaries on this work extant of which some also go back to the thirteenth century. Of these the one which seems to have been most widely used was that made by Petrus de Dacia in 1291. Of this commentary[2] five copies are mentioned by Enestroem[3] and five further by Curtze in the work cited. One Petrus de Dacia is given[4] as Rector of the University of Paris in 1326 and he may very well have been the author of this work. Petrus de Dacia wrote further a multiplication table from 1x1 up to 49x49,[5] a *Tabula de Loco Lunae,*[6] a Calendar and also a work "de calculo seu computo" which was probably on the computation of Easter.[7]

John of Halifax (Holywood, Holywod or Holiwod, Halifax, Holifax, Holybush, Holywolde Sacro-Busto or -Bosco, Sacrobosco, Sacrobuschus or Sacrobusto[8]) was born at Halifax, Yorkshire, studied at Oxford and later wandered to Paris where he lectured on mathenatics and astronomy. His most famous work was probably the *Sphaera mundi,* also known as *Libellus de Sphaera,* a treatise on astronomy which was used as a text-book in the Universities of the Middle Ages for several centuries.[9] He wrote further a *Computus Ecclesiasticus* and this *Algorismus,* which appears now to have been quite as popular from 1250-1550 as was the work on astronomy. The Algorismus was first published in Strassburg in 1488[10] and numerous editions followed. The text has been published also under the title *Tractatus de Arto Numerandi*[11] and *Oposculum de praxi numerorum quod Algorismum vocant.*[12] Copies of this work have been frequently attributed to other writers. The Plimpton Library MS. was ascribed to Mugling until Professor Smith made an examination of the text, while more recently an *Algorismus Doctrinalis* in the Hunterian Museum at Glasgow[13] is attributed to Michæl, Monk of Dover. As the opening and closing lines which are given are identical with lines in Sacrobosco's *Algorismus* it is seen to be the work of the same man, and this text is probably a student's copy made by this Michæl.

Nor is it to be suspected at all, that any attempt was made by the

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1). D. E. Smith, *Rara Arithmetica.*

2). Text by Max Curtze, l. c.

3). In publications of the Royal Danish Academy.

4). Potthast, Bibl. Historica Mediiaevi, who also mentions a Petrus de Dacia, Monachus Vistyensis, lector Skeningae, a. 1271-1275.

5). Enestrom, Bibl. Math., 1890, p. 32.

6). *A Catalog of the Manuscripts in the Library of the Hunterian Museum* in the University of Glasgow, begun by John Young and completed by P. H. Aitken, Glasgow, 1908. Other copies are in the British Museum, additional MSS., No. 35, 317, f. 1, in Oxford Univ. Lib. (Catalog of MSS., Ashmole, 1845), and in Paris.

7). Vossius, De Universae Mathesios Natura et Constitutione, Amstel., 1650.

8). Vossius, l. c., Smith, *Rara Arithmetica.*

9). Aschbach, J., *Gesch. d. Wiener Universitaet im erster Jahrhunderte ihres Bestehens.* Wien, 1865, p. 93.

10). Smith, l. c., pp. 31-33.

11). Halliwell, *Rara Mathematica.*

12). Paris, 1510, Clichtoveus edition.

13). Young and Aitken, l. c., pp. 388-389.

copyists of this work to take the credit for its composition. Many of the copies found are, doubtless, simply students' lecture notes which were, of necessity, as the use of this work was largely before the invention of printing, written out. Even a slight examination of the Columbia MS. and the Plimpton MS. shows such marked divergence in phraseology from the text published by Curtze (and from each other) that we are led to conclude that these were lecture notes largely for the personal use of the writer. The shorthand character of the writing, making their reading practically impossible in many places, further confirms this view.

This *Algorism* of Sacrobosco is of interest for the reason, also, that it is probably due to the extended use of this work that the appellation Arabic numerals became common. In two places there is mention of the inventors of this system. In the introduction it is stated that this science of reckoning was given out by a philosopher named Algus whence the name *Algorismus.* Petrus de Dacia recognizes the Arabic origin of the word Algos,[1] stating that this is the name of some Arabic philosopher. The Columbia manuscript gives also the derivation of *Algorismus* from *ars+rismos* which word means number. In the section on numeration reference is made to the Arabs as the inventors of this science.[2] While some of the commentators knew of the Hindu origin, e. g., Petrus de Dacia,[3] most of them undoubtedly took the text as it stood and so the Arabs were credited with the invention of the system.

In conclusion it may be well to state that this and the *Algorismus* of John of Lima are the two of the oldest works known which bear the name of al-Khowarazmi, for the work of Frater Sigsboto which was published by Curtze under the title *Uéber eine Algorismus-schrift des XII Jahrhunderts*[4] nowhere uses the term *Algorismus.* There is also the Saliminian Codex[5] of the end of the twelfth century (textual evidence of date, as it bears no date nor name of author), and the *Carmen de Algorismo*[6] of Alexander de Villadei of which some lines are quoted by Sacrobosco.

John Bale, who made an examination of English Libraries before 1548,[7] gives Marianus Scotus (1208-c. 1382) as the author of an Algorism. In the absence of any further knowledge of the MS. lost since, at least, 1748,[8] we cannot place much emphasis upon this statement. At the same time it is well to note that another *algorismus* of about this same time is

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1). "Quidam philosophus editit nomine Algus, unde et *Algorismus* nuncupatur," Curtze edition, l. c. p. 1. "Algorismus ab algos," Columbia MS. "Edidit Algus," also "Explicit Algorismus" at end, Plimpton MS.

2). "Sinistrorsum autem scribimus in hac arte more arabico sive indaico, huius scientiae inventorum," Curtze edition, p. 3. The Plimpton MS. omits the words "sive indaico."

3). "Non enim omnis numerus per quas cum que figuras in dorum repraesentatur," Curtze edition, p. 25.

4). *Abhandlungen zur Geschichte der Mathematik*, vol. VIII.

5). Index Britanniae Scryptorum, Joannes Balens, edited by R. L. Poole, Oxford, 1902, p. 287, "Ee collegio Martonensi, Oxon.

6). Halliwell l. c. This Carmen was widely used as shown by the large number of manuscript copies found in European libraries.

7). Index Brit., Script., 1548.

8). Tanner, Bibl. Britanno Hibernica, London, 1748,

credited to Guido Arezzo by Wailly[1] on the authority of the Benedictines, and also in the Nouveau traitè Diplomatique, in which the work is given as a computation on the sand table. This is the more likely inasmuch as several Arabic works written before 980[2] refer to calculation on a table, and to calculation without erasure. There is also a Computus of the XI century by Marianus in the Trinity College Library, bound with some arithmetic of the XII century.[3]

REVIEW OF A SIGNIFICANT TEXT IN CALCULUS.*

By N. J. LENNES, Massachusetts Institute of Technology.

This little book, entitled Elements of Differential and Integral Calculus, deserves more than a passing notice, especially because it is intended for a class of students differing considerably from those who now study elementary calculus in our colleges and universities. As stated in the preface, it is the result of a series of lectures delivered during the last six years to classes consisting chiefly of students of engineering and chemistry. In working over the lectures "it soon appeared," continues the preface, "that the mathematical knowledge which needs to be possesseed by a student before attempting the calculus is very much less than has been supposed. For example, the binomial theorem of algebra and the addition theorem of trigonometry are quite unnecessary. This book is written with the view of making the subject more easily and generally accessible than it has been hitherto. The principles of the differential and integral calculus ought to be counted as a part of the intellectual heritage of any educated man or woman in the twentieth century no less than the Copernican system or the Darwinian theory. In order to make a beginning no previous knowledge of mathematics is needed beyond the most elementary notions of geometry, a little algebra, including the law of indices and the definitions of the trigonometric funetions."

The general mode of presentation bears striking evidence of the influence of recent discussions of mathematical pedagogy. The consideration of each new process is preceded by concrete and simple problems in the solution of which the process is required. The simplicity of the problems themselves leaves the mind free to concentrate upon the one new thing offered for consideration, namely, the process of solution.

1). Elements de Paleographie, Paris, 1838, vol. I, pp. 711–716.

2). *M. Suter, Abhandlungen zur Geschichte der Mathematik, Fihrist,* pp. 37, 40, 41.

3). *Trinity College Library, Catalog of Manuscripts.*

*Elements of the Differential and Integral Calculus by A. E. H. Love. Cambridge University Press, pages 204+XIII.

The subject of coordinate geometry is introduced by such problems as the representation of the relation between the length of a spring and the weight of a body suspended from it, the relation between the readings of Centigrade and Fahrenheit thermometers, and the distances covered by a falling body in different intervals of time. The notion of "gradient" or "slope" is emphasized first in connection with the straight line and then in connection with the parabola obtained from the falling body problem. Numerous little exercises are interspersed for finding the gradient of straight lines and simple curves.

Chapter II deals with differentiation more formally under the following heads:

(a) Falling bodies.
(b) Speed of advancing body.
(c) Rates of change in general.
(d) Tangents to a curve.

Up to this point not a word has been said as to what is meant by "limit," though the word and the idea have been freely used. We now read (p. 21): "We shall be able to proceed more quickly afterward, and we shall be more certain that our work is correct, if we take a little time to think exactly what it is that we mean when we say that a function of h tends to a limit as h tends to zero." Then follows a beautifully simple exposition. Thus at every point the student is taken fully into the confidence of the author.

After the proofs of the formula for differentiating u^n for all rational values of n the second derivative is introduced and this is followed immediately by applications to tangents, approximations such as the use of $x^n+nx^{n-1}h$ for $(x+h)^n$, and more generally the use of $f(x)+hf'(x)$ for $f(x+h)$, maxima and minima including the use of $f''(x)$ to discriminate between a maximum and a minimum, and the mean value theorems.

Chapter IV (p. 55) first introduces integration. Practically throughout the book integration is regarded as the inverse process of differentiation and not as finding the limit of a sum. To find an area it is first asked "what is the derivative of the area?" and similarly for finding volumes, surfaces, lengths of areas.

Chapter VI, which deals with logarithms and exponential functions, is more complicated and difficult than the rest of the book, as are also the problems and applications of this chapter.

Chapter VII consists of a treatment of trigonometric functions. The proofs are unique inasmuch as little or no use is made of trigonometric formulas beyond the mere definitions of the functions. The theorems of this chapter are immediately applied to problems in oscillatory motion.

Chapter VIII deals with methods of integration and Chapter IX with applications to length of curves, curvature, and areas of surfaces of revolution.

In Chapter X (p. 157) the definite integral is considered as a limit of a sum, but not even here is it defined as such a limit. It is simply shown that the definite integral as previously defined is indeed the limit of a certain sum.

Chapter XI deals with centers of gravity, centers of pressure and moments of inertia. Each of these is regarded primarily as a limit of a sum.

The Appendix (pp. 179-204) contains formal proofs of certain propositions which in the text are made evident by rather informal discussion.

What is needed of trigonometry and analytic geometry is introduced as required. It is rather surprising to find that the 178 pages of the body of the text could surely not have been shortened by more than 25 pages if full knowledge of these subjects had been assumed.

For years past Klein in Germany, Perry in England, Moore in the United States, and hosts of others have been urging upon the mathematical fraternity the possibility and desirability of introducing the notions of the calculus at an earlier stage than is now done. In view of this, the little book under review is particularly significant. It gives a concrete means of judging whether such simplifications as would be demanded by distinctly less mature classes would be possible. The reviewer dare not undertake to say that this is an ideal book for very young classes, or even as well fitted to the American situation as we might reasonably expect were the book written expressly for us ; but barring a few minor features closely related to English usage where it differs from ours, and other features which are related to the particular class of students for which it was primarily developed, it seems that the grade of difficulty is adapted to the fourth year of our secondary schools or the first year in college.

Would not a one year course of college mathematics be vastly more interesting and instructive if based upon a book of this type than the present combination of trigonometry, college algebra, and analytic geometry? In a majority of our secondary schools we now have only three or at most three and a half years of mathematics. Would not such a course as this furnish admirable material with which to complete a full four years course?

Unfortunately the typography of the book is ineffective and the pages are monotonous in appearance. The page architecture is particularly bad. An important figure frequently occurs near the bottom of a right hand page with nearly all the discussion on the next page. But such minor defects aside, one may reasonably hope that this little volume points the way to a vitalizing of an important part of our mathematical curriculum.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

335. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

A person has \$1800 in notes payable \$18 monthly, bearing 10% interest. Find their present value if the interest is payable at the maturity of each note; also present value if interest is payable annually. [An actual business transaction.]

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

From the reading of the problem, it is supposed that the \$1800 is the actual amount to be paid, including interest.

I. Let x=present value.

$$\text{Then } x=\frac{18}{1.00\frac{5}{6}}+\frac{18}{(1.00\frac{5}{6})^2}+\frac{18}{(1.00\frac{5}{6})^3}+\ldots+\frac{18}{(1.00\frac{5}{6})^{100}}$$

$$=\frac{18}{(1.00\frac{5}{6})^{100}}[1+1.00\tfrac{5}{6}+(1.00\tfrac{5}{6})^2+\ldots(1.00\tfrac{5}{6})^{99}]$$

$$=\frac{18}{(1.00\frac{5}{6})^{100}}\left(\frac{(1.00\frac{5}{6})^{100}-1}{.00\frac{5}{6}}\right)=2160\left(1-\frac{1}{(1.00\frac{5}{6})^{100}}\right)$$

$$=2160(1-.436114)=1217.994, \text{ the number of dollars.}$$

II. In such transactions the interest for each year is calculated as though no payments were made. Hence,

$$x(1.10)^8(1.03\tfrac{1}{3})-216[(1.10)^7+(1.10)^6+\ldots+1](1.03\tfrac{1}{3})-72=0.$$

$$x(1.10)^8(1.03\tfrac{1}{3})=2160[(1.10)^8-1](1.03\tfrac{1}{3})+72.$$

$$x=2160\left(1-\frac{1}{(1.10)^8}\right)+\frac{72}{(1.10)^8(1.03\frac{1}{3})}.$$

$$x=\$1184.846.$$

336. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Evaluate the determinant

$$\Delta=\begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \ldots & a_n^2 \\ a_2^2, & a_3^2, & a_4^2, & \ldots & a_{n+1}^2 \\ a_3^2, & a_4^2, & a_5^2, & \ldots & a_{n+2}^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n^2, & a_{n+1}^2, & a_{n+2}^2, & \ldots & a_{2n-1}^2 \end{vmatrix}$$

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Subtract each row from the one which follows it, beginning with the last but one. Repeat the same operation, stopping at the second row. Keep repeating this operation, leaving out a row each time, until all the rows have been thus omitted; then if D=value of determinant and

$$\triangle^r a_s^2 = \triangle^{r-1} a^2_{s+1} - \triangle^{r-1} a_s^2,$$

we get

$$D=\begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \dots & a_n^2 \\ \triangle a_1^2, & \triangle a_2^2, & \triangle a_3^2, & \dots & \triangle a_n^2 \\ \triangle^2 a_1^2, & \triangle^2 a_2^2, & \triangle^2 a_3^2, & \dots & \triangle^2 a_n^2 \\ \cdot & \cdot & \cdot & & \cdot \\ \triangle^{n-1} a_1^2, & \triangle^{n-1} a_2^2, & \triangle^{n-1} a_3^2, & \dots & \triangle^{n-1} a_n^2 \end{vmatrix}$$

Repeating the same series of operations on the columns, we get

$$D=\begin{vmatrix} a_1^2, & \triangle a_1^2, & \triangle^2 a_1^2, & \dots & \triangle^{n-1} a_1^2 \\ \triangle a_1^2, & \triangle^2 a_1^2, & \triangle^3 a_1^2, & \dots & \triangle^n a_1^2 \\ \triangle^2 a_1^2, & \triangle^3 a_1^2, & \triangle^4 a_1^2, & \dots & \triangle^{n+1} a_1^2 \\ \cdot & \cdot & \cdot & & \cdot \\ \triangle^{n-1} a_1^2, & \triangle^n a_1^2, & \triangle^{n+1} a_1^2, & \dots & \triangle^{2n-2} a_1^2 \end{vmatrix}$$

If a_r^2 is a function of r of the pth degree in r, whose highest term has a coefficient unity, the quantities a_1^2, a_2^2, a_3^2, ... form an arithmetic series of the pth order.

If $p=n-1$ all the elements below the second diagonal vanish, while all those in it are equal to $(n-1)!$, and $D=(-1)^{n(n-1)/2}[(n-1)!]^n$.

If $m<(n-1)$, $D=0$.

These determinants have been called orthosymmetrical.

GEOMETRY.

361. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

$ABCD$ is a quadrilateral. The bisectors of A and C meet in O_1; those of B and D meet in O_2. Find the tangent of the angle between AD and O_1O_2 in terms of sines and cosines of A, D, $A+B$, and $A+D$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $ABCD$ be the quadrilateral. Produce AB, DC, and AD, BC until they intersect in E, F, respectively. Take ADE as the triangle of reference for trilinear coordinates.

Let $\gamma=0$, be the epuation to AD; $\beta=0$, the equation to AB; $\alpha=0$, the equation to DC; $l\alpha+m\beta+n\gamma=0$, the equation to BC.

Also let $P=\sqrt{[l^2+m^2+n^2-2mn\cos A-2nl\cos D-2ml\cos E]}$.

(1) $\alpha-\gamma=0$, bisects angle D.
(2) $l\alpha+m\beta+n\gamma-P\beta=0$, bisects angle B.
(3) $\beta-\gamma=0$, bisects angle A.
(4) $l\alpha+m\beta+n\gamma-P\alpha=0$, bisects angle C.
(1) and (2) intersect in

$$\frac{\alpha_1}{P-m}=\frac{\beta_1}{l+n}=\frac{\gamma_1}{P-m}=\frac{2\triangle}{(a+c)(P-m)+b(n+l)}=O_2.$$

(3) and (4) intersect in

$$\frac{\alpha_2}{n+m}=\frac{\beta_2}{P-l}=\frac{\gamma_2}{P-l}=\frac{2\triangle}{a(n+m)+(b+c)(P-l)}=O_1.$$

Equation to O_1O_2 is

(5) $\alpha\,(P-l)+\beta\,(P-m)-\gamma\,(P+n)=0.$

The angle between (5) and $\gamma=0$ is

$$\tan\phi=\frac{\sin A-\sin D+(l\sin D-m\sin A)/P}{1+\cos A+\cos D+(n-l\cos D-m\cos A)/P}.$$

But angle $(180^\circ-F)=(A+B)=$angle BC makes with AD.
$\therefore\sin(A+B)=(l\sin D-m\sin A)/P$;
$\cos(A+B)=(n-l\cos D-m\cos A)/P$.

$$\therefore\tan\phi=\frac{\sin A-\sin D+\sin\,(A+B)}{1+\cos A+\cos D+\cos\,(A+B)}.$$

362. Proposed by V. M. SPUNAR, M. and E. E., 3536 Massachusetts Avenue, N. S., Pittsburg, Pa.

Show that the focus of an ellipse may be regarded as an indefinitely small circle having double contact with the ellipse, the directrix being the chord joining the points of contact.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

A circle with its center at $(x_0,\ 0)$ any point of the major axis inside the evolute $[x_0<ae^2]$, and having for its radius the length of the normal which meets the axis in that point, is tangent to the ellipse at two points, say $(x_1,\ \pm y_1)$. From the equation of the normal to $b^2x^2+a^2y^2=a^2b^2$ at $(x_1,\ y_1)$ we find, since $a^2-b^2=a^2e^2$, $x_0=e^2x_1$; or $x_1=x_0/e^2$. Also the normal length is given by $N^2=(x_1-x_0)^2+y_1^2$, which reduces to $N^2=(1-e^2)\times(a^2-e^2x_1^2)=(1-e^2)(a^2e^2-x_0^2)/e^2$. Thus the circle has the equation

$$(x-x_0)^2+y^2=(1-e^2)(a^2e^2-x_0^2)/e^2 \tag{1}$$

Let x increase toward the limit ae; then while the intersections are imaginary for $x_0>ae^2$, the analytical conditions for tangency are still fulfilled. For $x_0=ae$ the right member of (1) vanishes and the circle becomes the focus, since $x=x_0=ae$, $y=0$ are the only real solutions of (1). The value of x_1 has then become ae/e^2 or a/e, the abscissa of all points in the directrix.

Also solved by V. M. Spunar, G. B. M. Zerr, J. Scheffer, and Levi S. Shively.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

168. Proposed by A. H. HOLMES, Brunswick Maine.

Find integral values for x, y, u, and v from the following:

$$uv-xy=25x+29y+29u+29v-112.$$
$$3v-5u+5y-x=102.$$
$$4y-3v=419.$$

Solution by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.

From (3), we have

$$v=\frac{4y-419}{3}\ldots(4),$$

which substituted in (2) and reduced will yield

$$u=\frac{9y-(x+521)}{5}\ldots(5).$$

Substituting (4) and (5) in (1), combining like terms, and we get

$$36y^2-7653y-x(19y-131)+326061=0.$$

$$\therefore x-\frac{36y^2-7653y+326061}{19y-131}=0, \text{ or } 19x-36y+\frac{140691y-6195159}{19y-131}=0, \text{ or}$$

$$361x-684y+140691-\frac{99277500}{19y-131}=0\ldots(6).$$

Hence, $99277500/(19y-131)$ must be an integer, and therefore $19y-131$ must be a factor of $99277500=2^2.3.5^4.7.31.61$. Thus

$$19y=131\pm(\alpha\,\beta\,\ldots),$$

where $\alpha, \beta, \gamma, \ldots$ are the six different prime factors 2, 3, 5, 7, 31, 61 (with repetition 2 and 5) combined in products of 1, 2, 3, ..., 10 letters at a time.

By actual calculation and checking we have found the following ten solutions:

$x = -3343,\ 12721,\ 320,\ 305,\ 7547,\ -2225,\ -710,\ -3215,\ -8495,\ -56950;$

$y = 2,\ 8,\ 29,\ 374,\ 4187,\ -1,\ -91,\ -1486,\ -4276,\ -29851;$

$u = 568,\ -2634,\ -116,\ 454,\ 5923,\ +337,\ -126,\ -2136,\ -6102,\ -42446;$

$v = -137,\ -89,\ -101,\ 319,\ 5473,\ -141,\ -261,\ -2121,\ -5841,\ -39941.$

Also solved by G. B. M. Zerr.

169. Proposed by R. D. CARMICHAEL, Princeton University.

Let $Q_n(x)=0$ be the equation whose roots are all the primitive nth roots of unity without repetition. In $Q_n(x)=0$ replace x by α/β, a fraction in its lowest terms, and clear of fractions. Let $Q_n(\alpha, \beta)$ represent the resulting first member. Set $n=mp$ where p is the largest prime factor of n. It is required to find all the integral values of α, β, m, p satisfying the following relations:

(1) $Q_{mp}(\alpha, \beta)=p$,

(2) $\alpha^m-\beta^m \equiv 0 \pmod{p}$.

One such solution is: $\alpha=2, \beta=1, m=2, p=3$. (See MONTHLY, Vol. XII, p. 89.)

[No solution of this problem has been received.]

170. Proposed by PATRICK WALSH, 1451 Annunciation Street, New Orleans, La.

The areas of rectangles A and B are respectively 15170 10/27 and 31230.3627. Find the sides and diagonal of each rectangle in exact or rational numbers.

Solution by B. F. FINKEL, Ph. D.

For A, let x and y be the dimensions of the field. Then

$$xy = 15170\tfrac{10}{27} = \frac{2^{14}.5^2}{3^3} \ldots (1), \text{ and } \sqrt{(x^2+y^2)} = d \ldots (2),$$

where d is the diagonal, which is to be rational. Solving (1) for y and substituting the value thus found in (2) and reducing, we have

$$\frac{\sqrt{(3^6x^4+2^{28}.5^4)}}{3^3y} = d.$$

Let $x=\dfrac{2^7.5}{3^2}z$. Then $d=\dfrac{2^7.5}{3^2z}\sqrt{(z^4+9)}$. Let $\sqrt{(z^4+9)}=z^2t-3$. Then

$$z^2=\frac{6t}{t^2-1}.$$

We must now find such values for t as will make $\frac{6t}{t^2-1}$ a perfect square. These values are 0, 1, 2, 3, $\frac{1}{2}$, $\frac{1}{3}$. For $t=2$, $z=2$,

$$d=\frac{2^6.5^2}{3^2},\ x=\frac{2^8.5}{3^2},\ \text{and}\ y=\frac{2^6.5}{3}.$$

For $t=3$, $z=\frac{3}{2}$, $d=\frac{2^6.5^3}{3^2}$, $x=\frac{2^6.5}{3}$, and $y=\frac{2^8.5^2}{3^2}$.

The values of 0 and 1 for t, give reciprocal limiting values for d, x, and y.

A similar treatment for B leads to the value, $z=2\sqrt{\frac{15t}{t^2-1}}$.

z is rational for $t=0$, 1, 4, $-\frac{1}{4}$, giving the values $z=0$, ∞, 4.

The values of x and y corresponding to $z=4$, are

$$x=\frac{3^2.19.179}{2^3\times5},\ y=\frac{3\times19.179}{5^2},\ \text{diagonal}=\frac{3^2.17.19.179}{2^3\times5}.$$

171. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve completely:

$$2x^2-1=y,$$
$$2y^2-1=z,$$
$$2z^2-1=w,$$
$$2w^2-1=x.$$

I. Solution by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

It will be shown that the only integral solution is $x=y=z=w=1$.

If $x=0$ or ±1, then $y^2=1$, $z=1$, $w=1$, so that the fourth equation gives $x=1$.

Next, let $x^2>1$. Then $y>x^2$, $z>y^2>1$, $w>z^2>1$, $x>w^2$.

Hence, $x>x^{16}$, which contradicts $x^2>1$.

Also similarly solved by G. B. M. Zerr, and S. G. Barton.

II. Solution by the PROPOSER.

Let $x=\cos\phi$; then $y=\cos2\phi$, $z=\cos4\phi$, $w=\cos8\phi$, $x=\cos16\phi$.

Since $\cos16\phi=\cos\phi$, we have, $\cos16\phi-\cos\phi=0$, which may be written

$$-2\sin\tfrac{17}{2}\phi.\sin\tfrac{15}{2}\phi=0.$$

If $\sin\frac{17}{2}\phi=0$, $\phi=\frac{2n\pi}{17}$. If $\sin\frac{15}{2}\phi=0$, $\phi=\frac{2n\pi}{15}$.

Taking $\phi=\frac{2n\pi}{15}$, we get seven roots, viz,

$$x_1=\cos\frac{2\pi}{15}=\cos 24^\circ=\tfrac{1}{8}[\sqrt{(30-6\sqrt{5})}+\sqrt{5}+1];$$

$$x_2=\cos\frac{4\pi}{15}=\cos 48^\circ=\tfrac{1}{8}[\sqrt{(30+6\sqrt{5})}-\sqrt{5}+1];$$

$$x_3=\cos\frac{6\pi}{15}=\cos 72^\circ=\tfrac{1}{4}(\sqrt{5}-1);$$

$$x_4=\cos\frac{8\pi}{15}=\cos 96^\circ=-\tfrac{1}{8}[\sqrt{(30-6\sqrt{5})}-\sqrt{5}-1);$$

$$x_5=\cos\frac{10\pi}{15}=\cos 120^\circ=-\tfrac{1}{2};$$

$$x_6=\cos\frac{12\pi}{15}=\cos 144^\circ=-\tfrac{1}{4}(\sqrt{5}+1);$$

$$x_7=\cos\frac{14\pi}{15}=\cos 168^\circ=-\tfrac{1}{8}[\sqrt{(30+6\sqrt{5})}+\sqrt{5}-1).$$

Taking $\phi=\frac{2n\pi}{17}$, we get eight roots. Gauss has shown how the complex roots of $x^{17}-1=0$ may be found by solving a chain of quadratic equations. If ω is a complex root of $x^{17}-1=0$, then

$$\omega=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}, \text{ and } \omega^{16}=\cos\frac{2\pi}{17}-i\sin\frac{2\pi}{17},$$

from which we have

$$\cos\frac{2\pi}{17}=\tfrac{1}{2}(\omega+\omega^{16})=x_8;$$

and similarly,

$$\cos\frac{4\pi}{17}=\tfrac{1}{2}(\omega^2+\omega^{15})=x_9,$$

$$\cos\frac{6\pi}{17}=\tfrac{1}{2}(\omega^3+\omega^{14})=x_{10},$$

$$\cos\frac{8\pi}{17}=\tfrac{1}{2}(\omega^4+\omega^{13})=x_{11};$$

$$\cos\frac{10\pi}{17}=\tfrac{1}{2}(\omega^5+\omega^{12})=x_{12};$$

$$\cos\frac{12\pi}{17}=\tfrac{1}{2}(\omega^6+\omega^{11})=x_{13};$$

$$\cos\frac{14\pi}{17}=\tfrac{1}{2}(\omega^7+\omega^{10})=x_{14};$$

$$\cos\frac{16\pi}{17}=\tfrac{1}{2}(\omega^8+\omega^9)=x_{15};$$

We have to solve the chain of equations:

$k^2+k-4=0.$ Roots k_1, k_2.
$l^2-k_1l-1=0.$ Roots l_1, l_2.
$l^2-k_2l-1=0.$ Roots l_3, l_4.
$4x^2-2l_1x+l_3=0.$ Roots x_8, x_{11}.
$4x^2-2l_2x+l_4=0.$ Roots x_9, x_{15}.
$4x^2-2l_3x+l_2=0.$ Roots x_{10}, x_{12}.
$4x^2-2l_4x+l_1=0.$ Roots x_{13}, x_{14}.

These fifteen roots together with the root $x=1$ make the sixteen roots. From symmetry, the sets of values for y, z, and w, are the same as for x.

PROBLEMS FOR SOLUTION.

ALGEBRA.

337. Proposed by I. M. CURTISS, Brooklyn, N. Y.

Three regiments move north as follows: B is 20 miles east of A; C is 20 miles south of B, and each marches 20 miles between the hours of 5 a. m. and 3 p. m. A horseman with a message from C starts at 5 a. m. and rides north till he overtakes B, then sets a straight course for the point at which he calculates to overtake A, then sets a straight course for the next point at which he will again overtake B, then rides south to the point where he first overtook B, reaching that point at the same time as C, namely 3 p. m. What uniform rate of travel enabled the messenger to do this?

338. Proposed by R. D. CARMICHAEL, Princeton University.

Prove that $\pi=3+\frac{1}{3}\cdot\frac{1}{1.2}-\frac{1}{5}\cdot\frac{1}{2.3}+\frac{1}{7}\cdot\frac{1}{3.4}-\frac{1}{9}\cdot\frac{1}{4.5}+\ldots$

339. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that if $a_1 \not< 2$ and $a_n=a_{n-1}^2-2$, $\frac{1}{a_1}+\frac{1}{a_1a_2}+\frac{1}{a_1a_2a_3}+\ldots$ $=\frac{1}{2}[a_1-\sqrt{(a_1^2-4)}]$.

GEOMETRY.

368. Proposed by G. I. HOPKINS, Professor of Mathematics and Astronomy, Manchester High School, Manchester, N. H.

It is required to construct the triangle having given, base, vertical angle, and ratio of its altitude to sum of the other two sides.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

370. Proposed by R. C. ARCHIBALD, Paris, France.

The trisectors of the angles of any triangle ABC are, in order, AF, AE, CE, CD, BO, BF. Show synthetically that D, E, F are the vertices of an equilateral triangle.

CALCULUS.

295. Proposed by C. E. FLANNAGAN, Wheeling, W. Va.

A hawk can fly v feet per second, a hare can run v' feet per second. The hawk, when a feet vertically above the hare, gives chase and catches the hare when the hare has run b feet. Find the length of the curve of pursuit. [Echols' *Differential and Integral Calculus*, page 253, Ex. 20.]

296. Proposed by C. N. SCHMALL, New York City.

Two currents C_1 and C_2 produce deflections ϕ_1, ϕ_2, respectively, in a tangent galvanometer. When is $(\phi_1-\phi_2)$ a maximum?

MECHANICS.

249. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A load P is supported by three strings of equal size attached at the vertices of a triangle sides a, b, c lying in a horizontal plane. The load is vertically under the centroid of the triangle at a distance h from it. Find the stresses in the strings.

250. Proposed by C. N. SCHMALL, New York City.

A smooth circular table is surrounded by a smooth vertical rim. A ball of elasticity e is projected from a point at the rim in a line making an angle ϕ with the radius through that point. Show that the ball will return to the starting point after the second impact if

$$\tan\phi=\sqrt{\frac{e^3}{e^2+e+1}}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

174. Proposed by B. KRAMER, Student, University of Pittsburg, Pittsburg, Pa.

Find a general solution of $x(x+a)=y^2$, a, x, and y being integers. Given a, required to find x to satisfy conditions.

NOTES AND NEWS.

The former students of Professor Bolza at the University of Chicago, desiring to express in some tangible form their love and esteem, presented to him a beautiful loving cup at a dinner in his honor on June third. The following is quoted from the letter of presentation:

"Those among the students of Professor Bolza who know him best are foremost in their appreciation of his unusual qualities, both as a lecturer and in awakening the spirit of research. Faithful in precept and inspiring in example, he has been a tower of strength at the University for nearly two decades, while students by the hundreds, after drawing from him mathematical inspiration and power, have gone forth to all parts of this country, many of them to occupy positions of responsibility and trust in our leading colleges and universities. As a genial friend whose hospitality we have all enjoyed, as an inspiring teacher whose peer we have seldom known, as a contributor to mathematical science whose reputation is established here and abroad, as the man who with Professors Moore and Maschke made the University of Chicago from its earliest days one of the foremost of mathematical schools, we pay respectful tribute to Professor Bolza, and wish him and Mrs. Bolza the widest usefulness and the greatest happiness in their new "old home" in Freiburg."

At the University of Tennessee Professor John B. Hamilton has a leave of absence for the coming year and will spend the time in advanced study at the University of Chicago.

At Cornell University Dr. Virgil Snyder and Dr. J. I. Hutchinson have been advanced to full professorships.

At the University of Wisconsin Dr. Max Mason has been appointed to a full professorship in mathematical physics.

At the University of Kansas Dr. C. H. Ashton and Dr. Van der Vrees have been advanced to associate professorships, and Professor J. W. Young of the University of Illinois has been appointed head of the department of mathematics.

The next meeting of the National Education Association will be held in Boston during the week of July 28, 1910. The program of the Mathematical Section has been arranged as follows:

Why Do We Study Mathematics; A Historical and Philosophical Retrospect by Prin. Thomas J. McCormack, Chairman LaSalle-Peru Township High School, LaSalle, Illinois.

Discussion: (a) The Practical Limitations of an Ideal Course in American Secondary Mathematics, and the Educational Waste or Economy

in the Proposed Sequences of Studies, by Principal John Shaw French, Morris Heights School, Providence, R. I.

(b). Preliminary Report of the "National Geometry Syllabus Committee," and its Practical Pedagogical Implications, by William Betz, East High School, Rochester, N. Y.

(c). Applied Problems and the Role of Formal Drawing in Secondary Mathematics, by William Breckenridge, Stuyvesant High School, New York City.

The committee on the teaching of mathematics to students of engineering, which was appointed at the joint meeting of the Chicago Section of the American Mathematical Society and Sections A and D of the American Association for the Advancement of Science, held in Chicago, December 30-31, 1907, will present its report at the meeting of the Society for the Promotion of Engineering Education to be held at Madison, Wisconsin, June 23-25, 1910. There will be extended outlines presented for discussion on the following topics: (1), Geometry and Mensuration; (2), Algebra; (3), Trigonometry; (4), Analytic Geometry; (5), Calculus; (6), Theoretical Mechanics; (7), Numerical Computation and the Solution of Equations. All mathematicians are urged to be present at that meeting and to take part in the discussions.

At the University of California, Dr. D. N. Lehmer has been advanced to an associate professorship in mathematics.

Dr. Arnold Dresden, of the University of Wisconsin, will spend the coming summer in Europe.

Professor C. Runge, Kaiser Wilhelm exchange professor of mathematics at Columbia University for the present academic year, recently delivered five lectures on graphical methods, at the University of Michigan. His lectures on this subject, delivered at Columbia University, are to be published in book form by Columbia University. These methods have many important applications in astronomy, physics and engineering. M.

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MATHEMATICS BEYOND THE CALCULUS.

By G. A. MILLER, University of Illinois.

Students of mathematics generally find that no option as regards the order of subjects is open to them until they have completed an elementary course in calculus. The prescribed road generally is as follows: arithmetic, algebra, geometry, trigonometry, analytic geometry, calculus. After completing an elementary course in calculus the student who expects to become a mathematician frequently finds an entirely different situation since it often becomes necessary for him to decide not only as regards the order of courses but also as to what courses should be selected from the rich offering.

He may perhaps be aided by the fact that there are only three grand divisions of mathematics according to some of the best authorities, even if these divisions do not have any distinct boundaries. They are commonly named as follows: arithmetic and algebra, analysis, geometry. As every mathematician should know something of each one of these great fields, the student will naturally plan to take some comprehensive courses in each of them. To divide the time at his disposal into three approximately equal parts would be easy enough, but to divide it into three parts with due regard to his tastes and ability, and to the men who may be offering the respective courses often becomes a very complex problem.

It is evidently desirable that the student should not decide very early on the particular small field where he hopes to be better informed than any one else. He will probably find many suitable fields without looking for them. In due time he has to make his choice and after it is once made he should bear in mind that serious and honest work is even more important than the choice of the field, for mathematics as a whole will profit by a harmonious development of all the various fields, and those who can appreciate work in but one field are excelled only by the mathematical butterflies in their pernicious influence.

One of the first things that such a student should do is to acquire some fairly intelligent notion as regards the extent of the mathematical literature. The fact that there are about a hundred thousand articles and thirty-five

thousand different books on mathematics, and that such a work as Mueller's *Vokabularium* gives about ten thousand technical mathematical terms may tend to a proper attitude of mind as regards knowing everything and a due appreciation of the need of collaboration. This collaboration naturally calls for leaders of broad sympathies and wide general knowledge, as well as for the more intense workers in limited regions. The rapid expansion of mathematical activity continually demands a greater variety of offorts and hence it offers to the young mathematician greater and greater freedom in selecting his field of usefulness.

Unfortunately it frequently becomes necessary for the student to elect a course about which he knows practically nothing except the name. It is difficult to avoid this. Definitions of a big subject have very little value until after the student has taken a course in the subject, and, strange to say, they frequently become of very little use even when a student knows considerable about the subject. Sometimes teachers, on being examined for public school positions, are asked to define such terms as algebra, and most of them are probably at about the right stage to give definitions. The little they know about the subject can readily be embodied in a definition. On further study, the subject appears to widen so much and to become so interwoven with others that a definition begins to appear hopeless. Hence, if definitions of such mathematical terms are to be given at all they should be given early.

While the student has frequently to begin some courses without much knowledge as regards the nature of the subject which is to be treated, he should aim to get at the sources of information on the subject as early as possible. To do this the collected works of eminent mathematicians are most important. The young mathematician cannot be too familiar with these original sources of information. The bibliographies on the periodical literature are also very helpful. The valuable services which the Royal Society of London Subject Index, the International Catalogue of scientific literature, the Jahrbuch ueber die Fortschritte der Mathematik, and the Revue Semestrielle des publications mathématiques are so well adapted to render should be fully understood, and the student should also acquaint himself, among others, with the following bibliographical aids: The great encyclopedia in German and in French, Hagen's Synopsis der hoeheren Mathematik, Pascal's Repertorium der hocheren Mathematik, Mueller's Fuehrer durch die Mathematische Literatur, Répertoire bibliographique, and Woelffing's Mathematische Buecherschatz. The first and last of these, respectively, cover the periodic and the non-periodic literature of the nineteenth century, the last being, however, very incomplete.

A very practical question is, what courses should be selected as first courses in each of the three great domains of mathematics mentioned above. In view of their extensive applications in other domains the following would be a very suitable set of first courses: theory of numbers, differential equa-

tions, and projective geometry. An almost equally important set of first courses in these three domains would be: theory of discrete groups, functions of a complex variable, and differential geometry. A third important set is higher algebra, functions of a real variable, and algebraic geometry. After completing one or two such general courses in each of these great domains, the student will probably have selected his special subject for thorough study and he may then wisely select more special courses. The number of such special courses that may be open to him will depend more upon the size of the faculty than upon the nature of the subjects. In the great encyclopedia the subject matter of each of the first two grand divisions of mathematics is presented under about thirty general headings, while that of geometry is placed under about forty such headings. As each of these headings is abundantly extensive for a course of lectures, we have here suitable names and material for about one hundred courses, without going outside of the field of pure mathematics and without general considerations of historic and pedagogic subjects.

From this great abundance of material and the tendency to give courses on very special subjects it is evident that the student who would acquire a comprehensive knowledge of mathematics has to do a large amount of reading on subjects which may not be closely related to the courses selected by him. In many of our universities this is especially true as regards the history of mathematics. If mathematicians wish to keep in contact with each other end especially with the larger circle of thinkers who are interested in the general intellectual development, they need a comprehensive view of the history of the various fundamental concepts of mathematics. Such a view is not generally acquired without much effort and the serious student will find that he can only secure a fairly satisfactory knowledge along such general lines by a persistent and wise use ot the moments which are not required for his regular work. Comprehensive historic knowledge will also tend to calmer judgment as regards the ephemeral waves of superficial interest in particular subjects.

We have thus far spoken only of pure mathematics. This is naturally the field of earliest interest as one cannot apply any unknown mathematics, and the applications are frequently more difficult than the considerations as regard ideal conditions of a simple nature. The young mathematicians should however bear in mind that not only ought mathematics be developed harmoniously as a whole, but this development should go hand in hand with the harmonious development of its applications. Sometimes these applications mean only a change of language, and there is no distinct boundary between pure and applied mathematics. Nevertheless, it is of the utmost importance that the mathematician should be able to express his results in the most effective and in the most useful language, and this demands a knowledge of the fundamental laws of the subjects to which mathematical thought may be applied. The wider and more thorough this knowledge the more

probable it will be that the results may be put into the most helpful form. It is in the border land between mathematics and various other sciences where one may reasonably look for the most useful developments, and next to these come the border lands between the various mathematical subjects themselves.

One striking feature of the mathematics beyond the calculus is that, as a rule, it is not arranged in such a carefully graded form as the earlier mathematics. If one considers how few new concepts enter into a course in elementary calculus, or a course in analytic geometry, and how much time and space is used in viewing these concepts from numerous points and how minute most of the thought steps are, one may wonder whether all this was really necessary. As a matter of fact, there is a great waste of time for the brightest student, but so many students of mediocre ability have to get over this ground that these details have been generally adopted.

On the other hand the average ability of the students who go beyond the elementary course in calculus is so very much higher that the thought steps are generally much longer. In fact, they are frequently of such lengths as to discourage the student. This discouragement generally lasts only a short time, and it is frequently replaced by keen enthusiasm, when the student begins to appreciate great thoughts rather than details of calculation, and when he learns to select view-points according to his own taste instead of slavishly following directions. It is true that this requires more time but it also brings better results and prepares the way for independent thought. As all mathematics consists in going from one thought to another near by and repeating this process, it is clear that the power of independent thought is the mathematician's El Dorado.

There is a tendency to speak of the relative importance of work in the different fields of mathematics. In this respect mathematics has much in common with mining. Some of the most enthusiastic reports are based on surface indications or upon developments which are totally inadequate to justify the reports. It is also true that some prospectors seem to lose the ability of passing calm judgments or speaking cautiously. The young mathematician should bear in mind that some of the largest fortunes have been acquired by working low grade ore by improved processes, and that the usual fate of the superficial prospector is not inspiring. In each of the three great grand divisions of mathematics there is in sight an inexhaustible supply of ore of various grades. Some prospecting work in these fields is, however, also highly desirable.

As great theories can be developed only by collaboration it is very important that mathematicians should publish their most useful results from time to time and that they should put them into best possible form. The young mathematician should, however, be very sure of the importance and newness of his results before he offers them for publication. In De Morgan's Budget of Paradoxes, page 4, there occurs the following good advice along this line:

"Most persons must, or at least will, like the lady in Cadogan Place, form and express an immense variety of opinions on an immense variety of subjects; and all persons must be their own guides in many things. So far all is well. But there are many who, in carrying the expression of their own opinions beyond the usual tone of private conversation, whether they go no further than attempts at oral proselytism, or whether they commit themselves to press, do not reflect that they have ceased to stand upon the ground upon which their process is defensible. Aspiring to lead *others*, they have never given themselves the fair chance of being led by *other* others into something better than they can start for themselves; and that they should first do this is what both these classes of *others* have a fair right to expect."

The student should bear in mind that it is necessary for the instructor to express opinions upon an immense variety of subjects and that some instructors seem to think it also necessary to assume the attitude that they know more about every subject under discussion than all the rest of the world put together. Students should not take such attitudes too seriously even if there may be a tendency among their fellow students to accept instructors at the instructors' estimates of their own abilities. Few habits are so harmful to a student as that of a slavish or even a worshipful attitude towards his instructors. On the other hand, he should seek information from many sources outside of the lecture room. If he does this he will become a more independent thinker and lay a much broader foundation for later development.

It should also not be assumed that all the mathematical talent of the world is concentrated in one locality. In mathematics we are still an idolatrous and thoughtless people, and one of the greatest evils we have to contend against is the worship of idols. Let all preach the gospel of only one *mathematics* and that all mathematical worship should be before his throne only. What one individual may do is insignificant when compared with the total development. Much of the time employed in mathematical pilgrimages to Mecca could be more wisely employed. These pilgrimages, however, have spread the contagion of enthusiasm and in this way they have done a great deal of good. The improved facilities as regards communication, especially by means of the mathematical periodicals, make every good library a suitable place for productive mathematical activity; and sound scholarly work should meet, and generally does meet, with a hearty reception on the part of the greatest mathematicians irrespective of the place where the work was accomplished.

In a very general way it may be said that arithmetic and algebra form the basal subjects of mathematics, that analysis is dependent upon arithmetic and algebra, and that geometry depends upon both of the other two grand divisions. The interdependence of these grand divisions is, however, becoming more and more pronounced, and there is a considerable part of geometry which has only a little in common with the other two divisions.

Hence a specialist in geometry may confine himself within just as narrow limits as a specialist in arithmetic and algebra may do. Since analysis is only algebra grown big, it is clear that it is more directly dependent upon the preceding grand division than geometry is.

As mathemathetics largely consists in going from one concept to a related concept, and as concepts which involve the most extensive intimate relations with others are the most helpful, it is clear that it is just as important at times to be able to overlook details as it is at other times to obtain interesting illuminating results from a study of details. For instance, Hamilton, in his lectures on quaternions, makes a distinction between operator and operandus which is unnecessary for applications and is also a hindrance to clear exposition.* One of the great advantages of abstract group theory rests upon overlooking the distinction between operator and operand in many applications. The distinction between a system of conjugate substitutions and a group of permutations as used by Cauchy and Serret is also of doubtful value. The mathematician must be able to leap from mountain top to mountain top as well as to dig out the gold from old river beds concealed by the work of geologic ages.

A CURIOUS MECHANICAL PARADOX.

By EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

Whether a paradox appears merely as a pest depends largely on the point of view. If the paradox lies in some well known and thoroughly accredited discipline such as elementary algebra, geometry, or mechanics, it is certainly a nuisance except in so far as it may be pedagogically instructive for the purpose of reinforcing, by its solution, the very principles which it would upset. To this class belong the demonstrations that $2=4$, usually dependent on a division by 0 or a carelessly placed sign in the extraction of a square root, and the proof that all triangles are isosceles or all angles right angles, dependent on incorrectly drawn figures, and finally the proposition that a rolling wheel must be lighter than one at rest owing to the resultant centrifugal force attributable to the rotation of the wheel about its instantaneous center. Such paradoxes are even a hindrance pedagogically rather than a help unless the fallacy involved can be made much clearer than the fallacious demonstration — a thing almost impossible to accomplish unless the student is so well gounded in the fundamentals of the science that he would not himself fall into like errors. On the other hand, when the paradox arises in a field which is not yet familiar even to specialists in it, a thorough

*E. Study, *Encyklopaedie der Mathematischen Wissenschaften*, vol. I, page 159.

examination of the paradox may be of first rate scientific importance. The attention which eminent men have paid to the contradictions developable in the theory of transfinite numbers and in the early discussions of the second law of thermodynamics might be mentioned as illustrations. It is doubtful if all the paradoxes connected with kinetic theory are yet cleared away from the statistical study of the interactions of matter and ether; and from the point of view of the new theory of relativity the Lorentz shortening of a body in its line of motion may easily take on the aspect of a paradox.

A curious and withal instructive example of the former kind of paradox has just been brought to my attention and seems worthy of solution not only for those who have been puzzled by it but for those who may never have thought of it. Here it is — problem, paradox, and solution.

PROBLEM. Suppose a tank of water with an efflux pipe, which terminates vertically, is resting on a frictionless plane when the water begins to flow. What is the motion of the system?

To make the problem as specific as possible and at the same time to remove any unnecessary lack of symmetry, let it be assumed that the efflux pipe 1234 starts from the center of the bottom of the tank, runs through a horizontal distance L and ends with a vertical section as in the figure. It is this vertical ending which introduces the paradox.

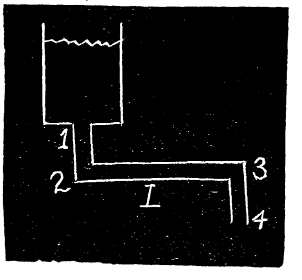


Fig. 1.

PARADOX. Let the water begin to flow out. As there are no horizontal external forces acting on the system, the center of gravity of the whole system must remain in the same vertical line. Hence the tank must slide to the left as the water is carried to the right. But as the tank slides to the left the water which leaves at 4 has a horizontal component toward the left and again, as there is no external horizontal force, the water will keep on indefinitely moving toward the left which is the same direction that the tank moves. Hence the center of gravity of the system will ultimately be found far to the left of its original position instead of in the same vertical line.

SOLUTION. The tank does start moving toward the left and a certain amount of water will move indefinitely to the left. But the tank will ultimately reverse its direction and move to the right and some of the water will move indefinitely to the right. And at all times the center of gravity of the system will be in the same vertical line and the system as a whole will never have any horizontal momentum.

That this is necessarily the solution is of course apparent as soon as one fixes his attention on the laws of momentum and abandons the attempt to follow intuitively and in detail the actual motion of the tank — but there would not be many paradoxes if one could only fix his attention rightly at the start. The really interesting thing about this particular paradox, how-

ever, is not its qualitative solution but the quantitive solution that follows immediately from applying the laws of mechanics to the system. Let v be the velocity of the water flowing in the tube 1234 and measured relatively to the tube. Let u be the velocity of the tank measured positively to the right. Let A be the cross-section and L the length of the tube 23. Let W be the amount of water in the tank and tubing at any instant and M the mass of the tank and tube. Then

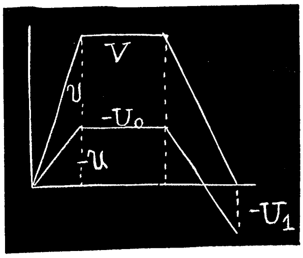


Fig. 2.

$$(M+W)u+AL(v-u)=(M+W-AL)u+ALv$$

is the total momentum at any time of the whole containing vessel and the water contained therein but not of the water which has flowed out; the total momentum of the water which has escaped is the negative of this. The rate at which water is flowing out is Av and as the velocity of tank is u, the rate of loss of momentum from the tank and tube is Avu. Hence

$$\frac{d}{dt}[(M+W-AL)u+ALv]=-Avu,$$

or

$$(M+W-AL)\frac{du}{dt}+AL\frac{dv}{dt}=0, \qquad (1)$$

since dW/dt, the rate of loss of water, is $-Av$. One may also write

$$(M+W)\frac{du}{dt}+AL\frac{dv'}{dt}=0, \qquad (1')$$

where $v'=v-u$ is the actual velocity of the water in the tube.

To tell just how the water would actually flow out would probably be a hydrodynamical problem incapable of solution in terms of elementary functions; but if any one of a number of reasonable assumptions be made as to the rate of change of v or v', the equations (1) or (1′) may be used to solve the motion of the tank. For instance, suppose that, when the efflux pipe is opened, the velocity v rises rapidly from 0 to a steady value V which is maintained for a while and then falls rapidly to 0 as the pipe is closed. As the rise of v is rapid, the coefficient $M+W-AL$ may be treated as constant during the rise. Hence $-u$ will rise from 0 to $-U_0$ where

$$-U_0=\frac{AL}{M+W_0-AL}, \ V \text{ and } W_0 \text{ is the initial value of } W.$$

Then the constant value U_0 will be maintained so long as V is maintained. Finally when v returns to 0 the coefficient $M+W-AL$ may again be regarded as constant and it is seen that u comes to the value

$$U_1 = -\frac{AL}{M+W_0-AL}V + \frac{AL}{M+W_1-AL}V, \text{ where } W_1 \text{ is the final } W.$$

This value of u is not 0 but positive, and indeed is necessarily so in order to counterbalance the negative momentum of the water which has escaped. The figure represents the variation of v and $-u$.

Other reasonable assumptions might be made, for instance Torricelli's law of efflux. Moreover the actual displacement of the tank could be obtained by integrating u. But sufficient has been said to show how readily the problem may be treated as far as its purely mechanical side is concerned. As it is none too easy to find really good and yet essentially different problems which require for their solution merely the fundamental principles of momentum, energy, and moment of momentum, it may not be amiss to point out that the above problem may be relieved of its hydrodynamical difficulties but otherwise preserved intact by considering a small number of beads on a wire lying in a vertical plane and consisting of a series of segments at different inclinations to the horizontal.

THE THEORY OF INVERSION AND THE QUADRATIC RECIPROCAL TRANSFORMATION.

By D. N. LEHMER, University of California.

1. The following discussion exhibits the theory of inversion in its proper light as a quadratic reciprocal transformation, and makes clear why inversion throws circles into circles but generally a curve of degree n into a curve of degree $2n$.

2. Consider a conic K and a point M. A correspondence between the points of the plane may now be set up as follows: To any point P make correspond the point P' of intersection of the line MP with the polar of P with respect to K. The point P' goes by this process into the point P again by the fundamental theorem in the theory of poles and polars: If the polar of P passes through P', the polar of P' passes through P.

If the point P moves along a straight line the point P' moves along a conic.

For P' is the locus of intersection of two projective pencils with centers at M and at the pole of the line upon which P moves. This conic thus

passes through a point M. A slight consideration will show also that it passes through the points A and B where the tangents from M to K touch K. For let P be on one of these tangents, then the line PM meets the polar of P in one of the points of tangency. Further, if P lies on the line AB the corresponding point P' is at M. More generally, we shall prove that

The transformation above described tranforms a curve of degree n into a curve of degree $2n$ which passes n times through the points A, B, and M.

For let C be a curve of degree n. It transforms into a curve C' of degree which it is proposed to determine. Cut across C' by a straight line a. Transform now again and C' goes back into C and the line a into a conic a'. The $2n$ points of intersection of a' with C correspond to the points of intersection of the line a and C'. Further, the curve C cuts the line AB in n points. The curve C' then passes n times through M. Similarly for points A and B.

The transformation is a little different from the ordinary transformation by which lines go into conics in that ordinarily points on a side of the singular triangle go into the *opposite* vertex. Here, while this is the case for the side AB, it is not the case for the other two sides, MA and MB.

The usual considerations apply for curves that go through one or more of the vertices A, B, and M of the singular triangle. Thus a conic generally goes into a quartic with three double points A, B, and M. If the conic goes through M the quartic degenerates into a straight line AB and a cubic through A, B and M with double points at M. If the conic goes through the points A and B the quartic becomes a conic and two straight lines. The two lines are MA and MB, the conic goes through A and B. Finally, if the conic goes through M, A, and B it transforms into a quartic made up of four straight lines one of which is the line that corresponds to that conic by the transformation; the other three being the sides of the singular triangle.

Consider now the special case where K is a circle and M is the center of that circle. Everything noted above still holds. If P moves along a line the point P' moves along a conic through M, A, and B. It is now seen that the points A and B are the points, usually denoted by I and J, where the circle K meets the polar of M, that is the line at infinity. If we define a circle as a conic through I and J, we have a transformation that throws a straight line into a circle. Moreover, a circle by this transformation goes into a circle, for as we have seen a conic through A and B goes into a conic through A and B. A conic generally goes into a quartic with double point at the center of K and double points at the circular points at infinity.

The metrical properties of the theory of inversion are now easily built up. It is not necessary to indicate the process further.

ON THE SOLUTION OF A SYSTEM OF LINEAR EQUATIONS.

By G. A. MILLER, University of Illinois.

A general system of m linear equations in n unknowns may be denoted as follows:

$$T=\begin{cases} a_{11}\ x_1+a_{12}\ x_2+\ .\ \ .\ \ .\ +a_{1n}\ x_n+k_1=0, \\ a_{21}\ x_2+a_{22}\ x_3+\ .\ \ .\ \ .\ +a_{2n}\ x_n+k_2=0, \\ .\qquad .\qquad .\qquad .\qquad .\qquad . \\ .\qquad .\qquad .\qquad .\qquad .\qquad . \\ a_{m1}\ x_1+a_{m2}\ x_2+\ .\ \ .\ \ .\ +a_{mn}\ x_n+k_m=0. \end{cases}$$

In the study of the solutions of this system the following two matrices are of especial importance.

$$A=\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & & . & & & . \\ . & & . & & & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{vmatrix}, \qquad B=\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} & k_1 \\ a_{21} & a_{22} & . & . & . & a_{2n} & k_2 \\ . & & . & & & . & \\ . & & . & & & . & \\ a_{m1} & a_{m2} & . & . & . & a_{mn} & k_m \end{vmatrix}.$$

Capelli exhibited the pedagogic advantage in employing, in the study of the system T, the concept of rank of these matrices, which are known, respectively, as the matrix and the augmented matrix of the system. The rank of a matrix is, according to Frobenius, the order of the largest non-vanishing determinant formed by elements of the matrix in order. The theorem proved by Capelli may be stated as follows: The necessary and sufficient condition that the system T is solvable is that the rank of the matrix of T is equal to the rank of its augmented matrix.* By solvable is meant that a set of n *finite* values may be assigned to the unknowns x_1, x_2, ..., x_n, so that each of the equations in T is satisfied.

When the system T is solvable it is also said to be consistent or compatible. If the system T is consistent and of rank r we may assign arbitrary values to at least one set of $n-r$ unknowns in this system so that after this is done it is possible to solve the resulting system. To each such $n-r$ arbitrary values there will correspond a single set of values for the remaining r unknowns, provided the $n-r$ unknowns to which arbitrary values were assigned were so selected that the rank of the matrix of the system is not diminished by omitting the coefficients of these $n-r$ unknowns from the matrix. As this interesting theory is so clearly presented in Bôcher's

*Capelli, *Rivista di Matematica*, Vol. 2 (1892), page 54. Capelli used the term characteristic instead of rank. The former of these terms is frequently employed in France and Italy. Cf. Pincherle, *Lezioni di algebra complementare*, 1909, page 92.

Introduction to Higher Algebra it does not appear necessary to enter into greater details here.

The main object of the present note is to consider the question: when can a given unknown in a consistent system of equations have only one value. To make this question perfectly clear we may consider the following system of three equations in three unknowns:

$$2x-y+2z=8,$$
$$4x-2y-z=-4,$$
$$6x-3y+z=-4.$$

It is evident that the rank of the matrix of this system is equal to the rank of the augmented matrix, each being 2. For every arbitrary value of x there is one and only one pair of values for y and z such that each of these three equations is satisfied. For instance, when $x=0$, y and z must have the value 0, 4 respectively; and when $x=1$ the values of y and z are 2, 4 respectively. Similarly, there is one and only one pair of values of x and z for every arbitrary value of y. On the contrary, z must always have the same value, namely, 4. We have therefore a system here in which the unknown z can have only one value although the system has an infinite number of solutions.

We are now in position to understand the following theorem:

The necessary and sufficient condition that a given unknown in a consistent system can have only one value is that the rank of the matrix of the system is diminished by omitting the coefficients of this unknown from this matrix.

That this condition is necessary follows directly from the general theory mentioned above; for, if the rank of this matrix were not diminished by omitting the given coefficients we could assign an arbitrary value to this unknown and solve the resulting system. Hence it remains only to prove that the given condition is also sufficient.

Suppose that the system T is consistent and that the rank of its matrix A is r, but that the rank of the matrix A' obtained by omitting the coefficients of x_α from A is less than r. The rank of A' must therefore be $r-1$ as the co-factor of at least one of these coefficients cannot vanish in a non-vanishing determinant of order r contained in A. As the matrix A' is of rank $r-1$ each of its rows can be expressed as a linear function of $r-1$ of these rows.* Hence we may replace the system T by a new system T' having the following form:

$$a_{1\alpha}x_\alpha + l_1 + k_1 = 0,$$
$$a_{2\alpha}x_\alpha + l_2 + k_2 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

*Cf. Bocher, *Introduction to Higher Algebra*, page 36.

$$\begin{array}{l} \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{r-1a}\, x_a + l_{r-1} + k_{r-1} = 0, \\ a_{ra}\, x_a + \phi_0(l_1, l_2, \ldots, l_{r-1}) + k_r = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{ma}\, x_a + \phi_{m-r}(l_1, l_2, \ldots, l_{r-1}) + k_m = 0, \end{array}$$

where $\phi_0, \ldots, \phi_{m-2}$ are the linear functions of $l_1, l_2, \ldots, l_{r-1}$.

The system T' is consistent since T is consistent and it must be of rank r since T has this rank. Hence it results that one and only one set of values for $x_a, l_1, \ldots, l_{r-1}$ will satisfy the first r equations. That is, x_a can have only one value in system T'. It can therefore have only one value in system T and the theorem in question has been established. From this theorem it results that each unknown in system T has either only one value or it has an infinite number of values whenever this system is consistent. In particular, a system of linear equations has always an infinite number of distinct solutions whenever it has more than one solution.

From the above it is evident that the language as regards solvability of a system of linear equations becomes much more concise by means of the concept of rank. Although this concept is comparatively new in mathematics it is of such fundamental importance that it should occupy a more prominent place in the courses in advanced algebra. It should be remembered that a thing can only appear simple when we can see clearly through it. In particular, the theory of linear equations appears simple only after an exhaustive study by means of such a powerful instrument as the concept of rank.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

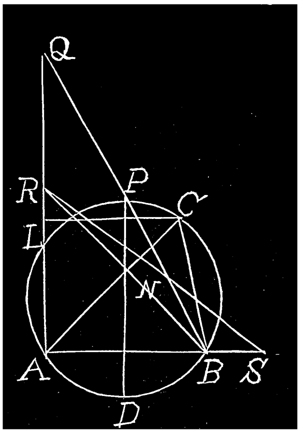
GEOMETRY.

363. Proposed by G. I. HOPKINS, Manchester, N. H.

Construct the triangle, having given, base, vertical angle, and difference between altitude and sum of the other two sides.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In what follows we assume that $AB = a =$ the given base; $\angle ACB = \angle C =$ given vertical angle; $p =$ difference of altitude and sum of other sides.



We further assume that $p>a<a\cot\frac{1}{2}C$. The solution would be just as simple with any other possible assumption.

On AB describe the segment ACB containing the given angle C. Draw the diameter PD perpendicular to AB. Also draw AQ perpendicular to AB. Draw BP meeting AQ in Q. With B as a center and a radius equal to p, describe an arc cutting AQ in R. With R as a center and a radius equal to $AQ-p$, describe an arc cutting AB produced in S. On SR measure off $SN=SA$. Then RN=altitude required. Take $AL=RN$, and draw CL parallel to AB, cutting the circle in C. Draw AC, BC. Then ACB is the required triangle. Let x, y, z be the sides BC, AC, and the altitude. Then $xy\sin C=az$, $x+y-z=p$, $a^2=x^2+y^2-2xy\cos C$.

$\therefore a^2+2xy(1+\cos C)=(p+z)^2$.

$\therefore z^2-2(a\cot\frac{1}{2}C-p)z=a^2-p^2$.

$\therefore z=a\cot\frac{1}{2}C-p-\sqrt{[(a\cot\frac{1}{2}C-p)^2-(p^2-a^2)]}$.

$\angle AQB=\frac{1}{2}C$, $AQ=a\cot\frac{1}{2}C$, $RS=a\cot\frac{1}{2}C-p$, $AR=\sqrt{[p^2-a^2]}$, $AS=\sqrt{[(a\cot\frac{1}{2}C-p)^2-(p^2-a^2)]}$.

$\therefore RN=z$. $\therefore$ The triangle ACB contains all the required parts.

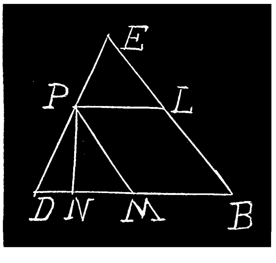
Also solved by J. Scheffer.

364. Proposed by R. C. ARCHIBALD, Providence, R. I.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length and directed to the opposite corner. ["Euclidean constructions" are particularly desired.]

Solution by C. N. SCHMALL, New York City, and J. SCHEFFER, A. M., Hagerstown, Md.

This construction cannot be effected by Euclidean geometry. This will be evident from the algebraic analysis of the conditions.



Let $MBLP$ be the given rhombus, and P the given corner. Also let DE be the required line, so that $DE=l$=given length. On AB drop the perpendicular PN. Now let $MB=BL=a$, $MN=c$, $DM=x$. Then we have, in $\triangle DMP$,

$$DP^2=DM^2+MP^2-2DM.MN=x^2+a^2-2cx.$$

Also, $DM^2:DP^2=DB^2:DE^2$;

that is, $x^2:(x^2+a^2-2cx)=(x+a)^2:d^2$.

$$\therefore d^2=\frac{(x+a)^2(x^2-2cx+a^2)}{x^2}=(x+a)^2\left(1-\frac{2c}{x}+\frac{a^2}{x^2}\right)$$

$$=x^2+2(a-c)x+2a(a-2c)+2a^2(a-c).\frac{1}{x}+\frac{a^4}{x^2}.$$

Also solved by G. B. M. Zerr.

CALCULUS.

286. Proposed by R. D. CARMICHAEL, Princeton University.

Solve the differential equation

$$\begin{aligned}&[a_0x^3+a_1x^2y+a_2xy^2+(a_0-a_1+a_2)y^3\\&\quad+a_3x^2+a_4xy+a_5y^2+a_6x+a_7y+a_8]dx\\&+[a_0y^3+a_1xy^2+a_2x^2y+(a_0-a_1+a_2)x^3\\&\quad+a_3y^2+a_4xy+a_5x^2+a_6y+a_7x+a_8]dy=0.\end{aligned}$$

I. Solution by W. W. BEMAN, Prosessor of Mathematics, University of Michigan, Ann Arbor, Mich.

Putting $x=s+t$, $y=s-t$, and afterward $t^2=w$, the equation takes the linear form

$$\frac{dw}{ds}+\frac{4(3a_0-2a_1+a_2)s+2(a_3-a_4+a_5)}{4(a_1-a_2)s^2+2(a_3-a_5)s+a_6-a_7}.w$$

$$=-\frac{4(a_0+a_2)s^3+2(a_3+a_4+a_5)s^2+2(a_6+a_7)+2a_8}{4(a_1-a_2)s^2+2(a_3-a_5)s+a_6-a_7}.$$

Or, putting $x+y=2u$, $xy=2v$, the equation takes the linear form

$$\frac{dv}{du}+\frac{4(3a_0-2a_1+a_{\cdot})u+2(a_3-a_4+a_5)}{4(a_1-a_2)u^2+2(a_3-a_5)u+a_6-a_7}.v$$

$$=\frac{8a_0u^3+4a_3u^2+2a_6u+a_8}{4(a_1-a_2)u^2+2(a_3-a_5)u+a_6-a_7}.$$

From each of these solutions it is obvious that the original equation has an integrating factor of the form $f(x+y)$.

II. Solution by E. B. ESCOTT, Ann Arbor, Mich.

Call the given equation $Mdx+Ndy=0$...(2).
If F is an integrating factor, we must have

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=\frac{N\frac{\partial F}{\partial x}-M\frac{\partial F}{\partial y}}{F}\text{...(3)}.$$

From symmetry of M and N, we may suppose F to be symmetrical in x and y. Suppose F is a function of $x+y$. Then

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z},$$

and equation (3) becomes an ordinary differential equation instead of a partial differential equation. (3) becomes when the values of M and N are substituted,

$$(4a_1-3a_0-3a_2)(x^2-y^2)+(a_4-2a_5)(x-y)=$$
$$-[(a_1-a_2)(x+y)^2+(a_3-a_5)(x+y)+a_6-a_7](x-y)\frac{\frac{dF}{dz}}{F},$$

or, dividing by $x-y$ and putting $x+y=z$,

$$(4a_1-3a_0-3a_2)z+a_4-2a_5=-[(a_1-a_2)z^2+(a_3-a_5)z+a_6-a_7]\frac{dF}{dz}.\frac{1}{F}.$$

The variables are separable, and we have

$$\frac{dF}{F}=-\frac{(4a_1-3a_0-a_2)z+a_4-2a_5}{(a_1-a_2)z^2+(a_3-a_5)z+a_6-a_7}dz=\left(\frac{a}{z-\alpha}+\frac{b}{z-\beta}\right)dz.$$

Then $F=(z-\alpha)^a(z-\beta)^b$, etc.

291. Proposed by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.

Integrate $\frac{dy}{dx}=ay^2+bx^m$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

This is the famous Riccati equation. It can be solved only under certain circumstances.

If $m=0$, we have $\dfrac{dy}{ay^2+b}+dx=0$, easily integrated. If $m=-2$, $\dfrac{dy}{dx}=ay^2+\dfrac{b}{x^2}$ becomes homogeneous, if y is replaced by z^{-1}. If $m=-4$, we put $y=\dfrac{1}{ax}+\dfrac{z}{x^2}$, then we have $\dfrac{dz}{az^2+b}+\dfrac{dx}{x^2}=0$, which can easily be integrated. When m has the form $-\dfrac{4n}{2n-1}$ or $-\dfrac{4n}{2n+1}$ the equation yields integrable forms, putting in the former case $x=1/u$, in the latter $y=1/z$.

This problem is fully treated in Johnson's *Differential Equations*, Chapter IX, and in Forsyth's *Differential Equations*, second edition, pages 170-176. A number of our contributors referred to these and other sources. Ed.F.

MECHANICS.

240. Proposed by S. A. COREY, Hiteman, Iowa.

A perfectly flexible wire rope weighing one pound per foot is suspended from the tops of two vertical supports 300 feet apart, one support being 30 feet higher than the other. One end of the rope is fastened to the top of the higher support, while 600 feet of the rope hangs vertically from the top of the lower support. Assuming that the rope is free to slide over the top of the lower support without friction, find the lowest point of that portion of the rope which is suspended between the supports. Also find the amount of work which must be performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope.

Discussion by F. H. SAFFORD, Ph. D., University of Pennsylvania.

This problem involves several points of theoretical interest which may be better treated as a general problem. The analysis will be carried to such a stage that the transcendental equations involved may be used to obtain numerical results by any convenient method of computation.

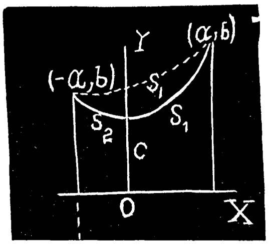
It is evident that between the supports the rope will hang in a catenary which, if referred to appropriate axes to be found, has the equation

$$y=c\cosh\frac{x}{c}. \tag{1}$$

Let the tops of the two supports be $(-a, b)$ and (a_1, b_1), and let the free end of the rope hang from the first point.

"The tention at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of the point" (Minchin's *Statics, or* Bowser's *Analytic Mechanics*). With this use of the term "ordinate" the X-axis must be the directrix of the catenary, and since in this problem the rope is of unit length, weights and lengths are numeric-

ally the same. Thus b is the length of the free end in the initial position of the rope. It is desirable to express the constants in the problem in terms of quantities which from the point of view of Mechanics would naturally be given, *i. e.*, the difference in level of the supports, the perpendicular distance between them and the length of the rope in the catenary. Thus with the notation used above let us write



$$\begin{aligned} a+a_1&=2d, \\ b_1-b&=h, \\ s+s_1&=l, \end{aligned} \tag{2}$$

where s and s_1 are the lengths of the left and right portions of the catenary measured from the vertex. Since the curve passes through the points $(-a, b)$ and (a_1, b_1) it follows from (1) that

$$\begin{aligned} b&=c\cosh\left(\frac{-a}{c}\right)=c\cosh\frac{a}{c} \\ b_1&=c\cosh\frac{a_1}{c}. \end{aligned} \tag{3}$$

The values of s and s_1 are found to be

$$s=c\sinh\frac{a}{c}, \quad s_1=c\sinh\frac{a_1}{c}. \tag{4}$$

From (2), (3), and (4),

$$\begin{aligned} h&=b_1-b=c\left[\cosh\frac{a_1}{c}-\cosh\frac{a}{c}\right]=2c\sinh\frac{a+a_1}{2c}\sinh\frac{a_1-a}{2c} \\ l&=s+s_1=c\left[\sinh\frac{a_1}{c}+\sinh\frac{a}{c}\right]=2c\sinh\frac{a+a_1}{2c}\cosh\frac{a_1-a}{2c}. \end{aligned} \tag{5}$$

$$\begin{aligned} h&=2c\sinh\frac{d}{c}\sinh\frac{a_1-a}{2c} \\ l&=2c\sinh\frac{d}{c}\cosh\frac{a_1-a}{2c}. \end{aligned} \tag{6}$$

$$\sqrt{(l^2-h^2)}=2c\sinh\frac{d}{c}, \tag{7}$$

$$a_1-a=2c\tanh^{-1}\frac{h}{l}. \tag{8}$$

Equation (7), though transcendental, is the most important result at this stage, as it may be shown that from it a single positive value of c is obtainable. From (8) and the first (2) followed by (3)

$$\left.\begin{aligned} a&=d-c\tanh^{-1}\frac{h}{l},\\ a_1&=d+c\tanh^{-1}\frac{h}{l},\\ b&=c\cosh\left[\frac{d}{c}-\tanh^{-1}\frac{h}{l}\right],\\ b_1&=c\cosh\left[\frac{d}{c}+\tanh^{-1}\frac{h}{l}\right]. \end{aligned}\right\} \tag{9}$$

Thus from (7) and (9) the positions of the vertex and directrix of the catenary may be found. [In the original numerical problem $b=600$, $b_1=630$, $2d=300$, and the elimination of a and a_1 from (3) and the first of (2) gives

$$c\left[\cosh^{-1}\frac{b}{c}+\cosh^{-1}\frac{b_1}{c}\right]=2d. \tag{10}$$

The value of c must be found from (10) and then a and a_1 from (3), after which l follows from (7) and the solution may proceed as in the general case.]

The work performed in raising the lowest point to make it coincide with the top of the lower support by exerting a pull on the free end of the rope may be computed from the difference in level of the center of gravity of the entire rope before and after the change in position. Assuming that the rope will admit of pushing as well as pulling, the work may even be negative. In taking a and a_1 positive the lowest point is between the supports, but when a is 0 the vertex is at the top of the lowest support and there is no work to be done. When a is not 0 the problem is possible provided s_1 is less than the distance between the tops of the supports, *i. e.*,

$$c\sinh\frac{a_1}{c} < \sqrt{(h^2+4d^2)}. \tag{11}$$

This inequality is the closest criterion to be imposed upon the given quantities l, h, and d, others arising earlier being omitted because now superfluous. The ordinate of the center of gravity of the rope in the two positions involves the determination, for the catenary, of

$$\int y ds = c\int \cosh^2\frac{x}{c} dx = \frac{c^2}{4}\sinh\frac{2x}{c} + \frac{cx}{2}. \tag{12}$$

Between the limits $-a$ and a_1, (12) gives

$$\int_{-a}^{a_1} y ds = \frac{c^2}{2}\sinh\frac{2d}{c}\cosh 2\left(\tan^{-1}\frac{h}{l}\right) + cd$$

$$= \frac{c^2}{2}\cdot\frac{l^2+h^2}{l^2-h^2}\sinh\frac{2d}{c} + cd. \tag{13}$$

Taking into account the free end of the rope, the center of gravity of the entire rope in the initial position may now be found.

For the final state of the rope a new c must be computed in terms of h, d, and a new l which is the original s_1, [vide (4)], $\bar{c}$ being given by

$$\sqrt{(s_1^2-h^2)} = 2\bar{c}\sinh\frac{d}{\bar{c}}. \tag{7a}$$

The equation of the curve is now

$$y = \bar{c}\cosh\frac{x}{\bar{c}}, \tag{1a}$$

(referred to new axes), and the positions of the new vertex and directrix may be computed as before. The limits in (12) are to be taken as 0 and $2d$, c becoming $\bar{c}$, hence

$$\int_0^{2d} y ds = \frac{\bar{c}^2}{4}\sinh\frac{4d}{\bar{c}} + \bar{c}d. \tag{13a}$$

The free end is now longer by s, and the ordinate of the center of gravity of the new system may be found as before, though measured from the new directrix. The difference in level of the two centers of gravity may now be

readily obtained. In numerical problems the task of solving the transcendental equations (7) or (10), and (7a) presents no practical difficulty.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

172. Proposed by H. C. FEEMSTER, York, Neb.

Show that $\frac{(nr)!}{n!(r!)^n}$ is an integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

By induction we get the following:
Suppose $r(n-1)!$ is divisible by $(r!)^{n-1}(n-1)!$. Now

$$\frac{(rn)!}{n!(r!)^n} \div \frac{r(n-1)!}{(r!)^{n-1}(n-1!} = \frac{(nr)!}{r\ (n-1)!} \times \frac{(n-1)\ !}{n!r!}$$

$$= \frac{nr(nr-1)(nr-2)\ldots\text{to } r \text{ factors}}{nr(r-1)!}$$

$$= \frac{(nr-1)(nr-2)\ldots\text{to } (r-1) \text{ factors}}{(r-1)!} = \text{an integer.}$$

Now $\frac{r!}{r!\ 1!}$=an integer, and hence $\frac{(2r)!}{(r!)^2 2!}$=an integer; $\frac{(3r)!}{(r!)^3 3!}$=an integer, and so on.

173. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find integral values satisfying the equation,

$$a_1^2+a_2^2+a_3^2+\ldots+a_n^2=d^4.$$

I. Solution by FRANK LOXLEY GRIFFIN, Ph. D., Williams College.

One set of solutions may be obtained by putting $a_1=df_1$, $a_2=df_2$, etc., which reduces the problem to that of finding a sequence of n integers, the sum of whose squares is a perfect square. Or, geometrically, we seek $n-1$ right triangles whose sides are all integers, and the hypotenuse of each being one leg of the next.

I. This is readily accomplished for $n=2$ by recalling that, if (p^2-q^2) and $2pq$ are legs, where p and q are integers, the hypotenuse is also an integer, (p^2+q^2). Let f_1 be *any odd integer* (>1) and take $p-q=1$ and $p+q=f_1$, so that $p=\frac{1}{2}(f_1+1)$, $q=\frac{1}{2}(f_1-1)$, both integers. Thus the sides

of the right triangle are f_1, $\frac{1}{2}(f_1^2-1)$, and $\frac{1}{2}(f_1^2+1)$. Or, calling the second f_2 and the third d, we have $d^2=f_1^2+f_2^2$.

II. Let this value of d for $n=2$ be denoted by d_2, and observe that d_2 is *odd*. For f_1=say, $2m+1$ (being odd), whence $d_2=2m^2+2m+1$, odd. Thus d_2 may be treated precisely as was f_1, giving $f_3=\frac{1}{2}(d_2^2-1)$, $d_3=\frac{1}{2}(d_2^2+1)$, or $d_3^2=d_2^2+f_3^2=f_1^2+f_2^2+f_3^2$.

Since again d_3 is odd, the process may clearly be continued indefinitely. The successive values of f and d may be found from: $2f_{n+1}=d_n^2-1$, $2d_{n+1}=d_n^2+1$. The following table gives the values up to $n=6$ if we begin with f_1+3:

f_n	3	4	12	84	3612	6526884
d_n		5	13	85	3613	6526885

To obtain the proposed a's for any value of n, merely multiply each f (up to f_n) by d_n. Thus, for $n=3$,

$$39^2+52^2+156^2=13^4.$$

[We might also start with any *even* integer (>2), and either (A) form a sequence all terms of which are even multiples of the terms obtaining by beginning with any odd factor of f_1, or (B) let $f_1=2pq$ and take for p, q *any* factors whose product is $\frac{1}{2}f_1$. Thus for $f_1=14$:

$$p=7,\quad q=1,\quad f_2=48,\quad d_2=50,$$
$$p_2=25,\quad q_2=1,\quad f_3=624,\quad d_3=626,\text{ etc.}]$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Dr. Martin's solution can be extended to suit this requirement.

Let $x_1^2+x_2^2+x_3^2+\ldots+x_n^2=c^2$.

Then $c^2-n^2=(c-n)(c+n)=x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2$.

Let $c-n=b$, then $c+n=(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2)/b$.

$\therefore x_n=\frac{1}{2}(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2-b^2)/b$.

$c=\frac{1}{2}(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2+b^2)/b$.

Multiplying through by c^2 after substituting, we get

$$(cx_1)^2+(cx_2)^2+(cx_3)^2+\ldots+[(c/2b)(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2-b^2)]^2$$
$$=[(1/2b)(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2+b^2]^4.$$

$$\therefore (4b^2cx_1)^2+(4b^2cx_2)^2+(4b^2cx_3)^2+\ldots[2bc(x_1^2+x_2^2+x_3^2+\ldots$$
$$+x_{n-1}^2-b^2)]^2=(x_1^2+x_2^2+x_3^2+\ldots+x_{n-1}^2+b^2)^4.$$

Let $n=7$, $b=3$, x_1, x_2, x_3, ...=1, 2, 3, ..., 6. Then $c=\frac{50}{3}$.

$\therefore (600)^2+(1200)^2+(1800)^2+(2400)^2+(3000)^2+(3600)^2+(8200)^2=(100)^4$, or $6^2+12^2+18^2+24^2+30^2+36^2+82^2=10^4$.

Also solved by the Proposer.

PROBLEMS FOR SOLUTION.

ALGEBRA.

240. **Proposed by W. J. GREENSTREET, M. A., Editor The Mathematical Gazette, Stroud, England.**

Let $S_{n-1}=1^{n-1}+2^{n-1}+3^{n-1}+\ldots+(n-1)^{n-1}$. Find n if $S_{n-1}-(n-1)$ is a multiple of n^2.

341. **Proposed by O. L. CALLICOTT, Gettysburg, South Dakota.**

Prove that the sum of the series, $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots$ to infinity $=$ the sum of the series $\frac{1}{2}.1+\frac{1}{2^2}.\frac{1}{2}+\frac{1}{2^3}.\frac{1}{3}+\frac{1}{2^4}.\frac{1}{4}+\ldots$ to infinity.

342. **Proposed by E. B. ESCOTT, Ann Arbor, Michigan.**

Prove that $\frac{1}{1.2.3.4}+\frac{1}{5.6.7.8}+\ldots=\frac{1}{4}\log 2-\frac{1}{24}\pi$. [Hobson's *Plane Trigonometry*, page 348.]

GEOMETRY.

371. **Proposed by W. S. HUGHES, Student, Williams College.**

A right circular cone is cut by two parallel planes, one passing through the vertex, and each cutting both nappes. Are the straight lines which constitute the first section parallel to the asymptotes of the hyperbola forming the other section?

372. **Proposed by DANIEL KRETH, Oxford, Iowa.**

In the right triangle ADE right angle A, are given: $AB=9$, $BC=280$, $CD=35$, angle $AEB=$angle CED; required the distance AE.

373. **Proposed by S. LEFSEHETZ, East Pittsburg, Pa.**

Draw a circle passing through a given point and orthogonal to two given circles.

CALCULUS.

297. **Proposed by PROF. L. C. WALKER, Socorro, New Mexico.**

A square hole $2s$ on a side is cut through an ellipsoid, axes $2a$, $2b$, $2c$, the axis of the hole coinciding with the axis $2c$ of the ellipsoid. Find (1) the volume, and (2) the surface removed.

298. **Proposed by C. N. SCHMALL, New York City.**

Prove, by calculus, that if two regular polygons have equal perimeters, that which has the greater number of sides has the greater area.

399. **Proposed by JOSEPH V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.**

A cow is pasturing outside a circular field containing 10 acres. What length of rope will allow her to graze over exactly two acres?

MECHANICS.

351. Proposed by J. G. ROSE, B. A. (Oxion), Mt. Angel College, Oregon.

ABC is a uniform triangular lamina of weight $3W$ such that $AB=2AC$. A particle of weight W is attached to it at C. Show that if the lamina be suspended from angle A, it will rest with AB and AC equally inclined to the vertical.

352. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Assuming the earth's orbit to be a circle, if a comet move in an ellipse around the sun, in the same plane as the earth, with its perihelion nearer the sun than the earth, how long will it remain within the earth's orbit? Apply this to Halley's comet with the above assumptions, and supposing the eccentricity of its orbit to be 0.971733. Find its periodic time.

353. Proposed by W. J. GREENSTREET, M. A., Editor, The Mathematical Gazette, Stroud, England.

R_1 and R_2 are ranges on a horizontal plane of particles projected with given velocity from A on the plane to pass through B. Show that $a(R_1+R_2)-R_1R_2=\frac{a^4}{c^2}$, where $c=AB$ and a is the horizontal projection of AB.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

175. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Nebraska.

Show (a) that $[(2n)!]/[(n+1)!n!]$ is an integer, and (b) that $[(2a)!(2b)!]/[a!b!(a+b)!]$ is an integer.

AVERAGE AND PROBABILITY.

206. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A point O is taken at random in a triangle. What is the probability that if three other points are taken at random, one shall lie in each of the three triangles AOB, BOC, and COA? [Williamson's *Integral Calculus*, page 412, ex. 85.]

207. Proposed by JOSEPH V. COLLINS, Ph. D., State Normal School, Stevens Point, Wisconsin.

A formula for simple interest is, $i=prt/12$. If r is 5 or 7, what is the chance that the denominator will cancel out entirely into either p or t or both (where t is expressed in months and tenths of a month, written decimally with usually a common fraction after tenths). It is assumed that a factor of 12 can cancel into t if the integral part is divisible by 2, 3, or 4. Thus, to find the interest at 5% on $729.44 for 1 year, 9 months, 28 days, we write

$$1.8236\times5\times\frac{21.9\frac{1}{3}}{12}=?$$

NOTES AND NEWS.

The August-September number of the MONTHLY will be published September 28th.

At the University of Pennsylvania hereafter the chairman of each department of instruction will be elected annually by the department. For the year 1910-1911, Professor George Egbert Fisher has been chosen chairman of the Department of Mathematics in the Graduate School, and Professor Isaac J. Schwatt chairman of the Mathematical Deparmennt in the College.

In the Department of Mathematics at the University of Pennsylvania Assistant Professor G. H. Hallett has been promoted to a professorship and Dr. M. J. Babb, Dr. G. G. Chambers and Dr. O. E. Glenn have been promoted to instructorships.

At Purdue University Professor Erastus Test will retire at the close of the present academic year, and Professor Jacob Westlund has been promoted to a full professorship in mathematics.

The series of articles on the teaching of collegiate mathematics, which was closed for the present year with the paper by Professor Bailey in the last issue, is appropriately supplemented in this number by Professor Miller's paper on "Mathematics Beyond the Calculus," which is intended primarily for students but which should prove equally valuable to many teachers. There are already indications that this series of papers will bear fruit in further discussions, through these columns, of many important phases of the educational side of mathematics, which deserves careful reconsideration.

Mr. T. H. Hildebrandt, who is instructor in mathematics at the University of Michigan, has just completed his work for the Doctorate at the University of Chicago by presenting his thesis on "A contribution to the Foundations of Frechet's Calcul Fonctionnel." The degree was conferred at the convocation on June 14, 1910.

"The Absolute Minimum of a Definite Integral in a Special Field" is the title of the thesis presented by Mr. E. J. Miles for the Doctorate at the University of Chicago. Dr. Miles has been appointed to an instructorship at Cornell University.

Miss Marion B. White has received an appointment as assistant professor of mathematics at the University of Kansas. Miss White was formerly instructor at the University of Illinois, which position she left to pursue graduate work at the University of Chicago.

Dr. Elizabeth R. Bennett has been appointed instructor in mathematics at the University of Nebraska, and not at the University of Kansas as previously announced in these columns.

Mr. J. O. Pyle, professor of mathematics at Howard Payne College, Texas, has resigned to pursue advanced work at the University of Chicago.

Dr. R. C. Archibald, of Brown University, has spent the year 1909-10, on leave of absence, studying in Paris. He returns next year as assistant professor of mathematics.

Mr. H. C. Currier, instructor in mathematics at Brown University, has leave of absence for the coming academic year. He will spend the year in study in German universities.

Dr. Phillips, instructor in mathematics at the Massachusetts Institute of Technology, is attending lectures at the University of Chicago during the present Summer.

Professor F. S. Woods, of the Massachusetts Institute of Technology, will resume his duties in the autumn, after a year's absence in Europe.

Mr. C. J. West, fellow in mathematics in the University of Illinois, has been appointed instructor in mathematics in the University of Ohio.

Mr. Benjamin Carter, of the Massachusetts Institute of Technology, has been appointed instructor in mathema,ics at Colby College, Maine.

Professor G. A. Miller will teach in the summer session at the University of Illinois.

Dr. O. D. Kellogg, of the University of Missouri, has been promoted to a full fellowship in mathematics.

Mr. R. P. Baker, instructor in mathematics at the University of Iowa, has been promoted to an assistant professorship.

Professor E. H. Moore will spend six weeks in Colorado and return to lecture at the University of Chicago during the second term of the summer quarter.

BOOKS.

First Course in Algebra. By Herbert E. Hawkes, Assistant Professor of Mathematics in Yale University, William A. Luby, and Frank C. Touton, Instructors in Mathematics, Central High School, Kansas City, Mo. 12mo, cloth, vii+334 pages, list price $1.00. Boston and Chicago: Ginn & Co.

"The 'First Course in Algebra' is designed for the first year's work. The topics considered have been strictly limited to those which belong primarily to study in the first year. Many helpful suggestions, the fruit of the widespread discussion of mathematical teaching which has marked the progress of the past ten years, are embodied in the book.

Difficult exercises have been avoided. Those given are new, varied, graded with extreme care, and amply sufficient to develop the essentials of elementary technic. The principles, so far as possible, are developed from the student's knowledge of arithmetic.

An abundance of typical solutions are given, and rules based on them have been carefully stated in full. Wherever practicable, suitable methods of checking are illustrated."

Plain Trigonometry. By Fletcher Durel, Ph. D., Head of the Mathematics Department of the Lawrenceville School. 8vo, cloth. 184+114 pages of tables. Price, $1.25. New York: Charles E. Merrell Co.

Among the special features claimed for this book, are the following: Under each case in the solution of triangles, examples are given in which the degrees are divided sexagessimally and decimally, the latter division of the degree meeting the new requirements of Harvard, Yale, and Princeton. A chapter given to the treatment of logarithms and other properties; a chapter in which the application of trigonometry are reduced to a system; and the enlivening of the subject matter by a chapter on the history of Trigonometry.

The author, in the arrangement of his logarithmic work adopted the form of tabulation used in the U. S. Navy Department and by engineers in general. This, to my mind, is of doubtful pedagogical value, since it omits the equality signs and thus breaks the solution into fragments instead of presenting it in concise symbolic sentences.

The book is neatly written and systematically arranged. The publishers have done their part to make the work attractive to both teacher and student. F.

Factor Table for the First Ten Millions. By D. N. Lehmer, Washington, D. C., Carnegie Institute of Washington, 1909. xiv+476 pages.

The chief factor tables published hitherto are those by Burckhardt for the first three millions, Glaisher for the next three millions, and Dase and Rosenberg for the seventh, eighth and ninth millions; they contain no less than 246 errors listed by Lehmer. In view of the difficulty of obtaining copies of certain of these tables, the appearance of Lehmer's table in a single volume will be very welcome to arithmeticians. The greater accuracy of Lehmer's tables rests not merely upon the essential improvements in its construction but chiefly upon the fact that it was constructed independently of the earlier tables, so that errors could be eliminated by noting discrepancies with the earlier tables, as detected by a comparison, entry for entry, made five times. Finally a very important check was made by counting the primes within various limits and comparing these counts with the numbers computed by Bertelsen by a modification of Meissel's method, the results tallying in every case. We congratulate Dr. Lehmer on this happy conclusion of his formidable task and express our gratitude to the Carnegie Institution for its aid in the preparation and publication of this monumental table. L. E. DICKSON.

Elementary Experimental Mechanics. By A. Wilmer Duff, M. A., D. Sc., (Edinburgh) Professor of Physics in the Worcester Polytechnic Institute, Worcester, Mass. 12mo. Red cloth, ix+267 pages. Price, $1.60 net. New York: The Macmillan Co.

This book is an attempt to make mechanics as attratctive as possible by a suitable collection of laboratory exercises. Among them are the Addition of Displacements, Path of Projectile, Motion of Pendulum, Simple Harmonic Motion, Force and Acceleration, Conical Pendulum, Friction, Moment of Force, Kinetic and Potential Energy, Torsion Pendulum, the Gyroscope, Torsion of a Wire, Flexure of a Bar, Archimedes' Principle, etc., etc. The apparatus to cary out the experiments is of the simplest design. The explanation and discussion is clear and fairly exhaustive. The author tells us that while the exercises have been chosen with a view to elucidate principles, yet the necessity of precise measurement has been kept in mind. The aim of the course, he says, is to stimulate reflection on concepts and principles and the value of each exercise is in proportion to the importance and number of physical ideas which must be considered in performing the exercise.

The book is neat and attractive in appearance, and its contents will be very valuable to those teachers who can avail themselves of its use. F.

Elementary Projective Geometry. By A. G. Pickford, Sometime Scholar of St. John's College, Cambridge, Headmaster of the Holme Grammar School, Oldham. 8vo. Red cloth, xi+256 pages. Price, 4s. Cambridge, Eng.: The University Press. New York. U. S. A.: G. P. Putnam's Sons.

The author of this work has arranged in orderly sequence the elementary propositions of Plane Projective Geometry, assuming on the part of the student only a knowledge of the first six books of Euclid or their equivalent. Proceeeing from the projective unit, the cross-ratio of four colinear points or four concurrent lines, the author takes up the discussion of projective rows and pencils, and the involution of six points or lines. He then deduces the properties of the curves of the second degree, the properties of the polars, and inscribed and circumscribed polygons, with construction of conics satisfying five given conditions.

The book ends with a brief consideration of Polar Reciprocation and Plane Homology, also brief notes on Projection in Space. In keeping with all work done by the University Press, the typography and mechanical execution of the book is first class. F.

An Introduction to Physical Science. By Frederick H. Getman, Ph. D., Associate in Chemistry, Bryn Mawr College, formerly Carnegie Research Assistant in Physical Chemistry, Johns Hopkins University. 12mo, cloth. v+257 pp. 129 figures. Price, $1.50 net. New York: John Wiley & Sons.

The author of this work, believing that many of the difficulties encountered by the beginner in chemistry may be traced to his unfamiliarity with the metric system, with the laws of elementary mechanics, or with simple thermal and electrical phenomena, has prepared this book to meet the needs of this class of students.

The work thus embodies a series of simple experiments demonstrating the more important phenomena in heat, light, electricity, and magnetism. F.

An Introduction to the Science of Radio-Activity. By Charles W. Raffety. With Illustrations. 8vo, cloth, xii+208 pages. Price, $1.25 net. New York: Longmans, Green & Co.

This book attempts "to give a concise and popular account of the properties of the radio-active elements and the theoretical conceptions which are introduced by the study of radio-active phenomena." The author has succeeded very well in giving the reader a fair idea of the main discoveries made during the last decade in this most interesting field of physical research.

The book is divided into three parts: The descriptive, containing eight chapters; the theoretical, containing eight chapters; and the practical, containing five chapters.

Every chapter is full of interest. F.

Problemes et Exercises de Mathématiques Générales. E. Fabry, Professeur a L'Universite de Montpellier. 8vo. Paper, 420, pages. Price, 10fr. Paris: A. Hermann & Sons.

In this book is to be found a fine collection of problems in Algebra, Analytic Geometry, Analysis, and Mechanics, covering the first 80 pages. The remainder of the book is devoted to the solution of the problems and their discussion. It is a good book from which to draw problems for supplementary exercises. F.

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ON THE CLASSIFICATION OF SYSTEMS OF LINEAR EQUATIONS.

By ARTHUR RANUM, Cornell University.

In a recent paper in THE AMERICAN MATHEMATICAL MONTHLY,* "On the Solution of a System of Linear Equations," Professor Miller has proved that in all the solutions of a system of consistent linear equations a particular unknown x_i will have the same value, that is, will satisfy an equation of the form $bx_i+l=0$ $(b\neq 0)$, if and only if the omission of the coefficients of x_i from the matrix of the system reduces the rank of the matrix. This theorem immediately suggested to me a generalization in which several unknowns are involved instead of only one, and I soon found that the more general theorem so derived is a sufficient basis for, and naturally gives rise to, a complete classification of systems of consistent linear equations, not only with respect to the rank of their matrices, but also with respect to the rank of the partial matrices obtained by omitting the coefficients of one or more unknowns. In this note I wish to state and prove the theorem, explain the resulting classification, apply it to one or two simple cases, and point out its geometrical significance.

REDUCTION IN RANK.

1. Consider any system T of m linear equations in $n=s+t$ unknowns, as follows:

$$T\begin{cases} a_{11}x_1+\ldots+a_{1s}x_s+a_{1,\,s+1}x_{s+1}+\ldots+a_{1n}x_n+k_1=0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{m1}x_1+\ldots+a_{ms}x_s+a_{m,\,s+1}x_{s+1}+\ldots+a_{mn}x_n+k_m=0. \end{cases}$$

Theorem. *A necessary and sufficient condition that in all the solutions of a system of consistent linear equations the values of any particular set of unknowns* $x_{s+1}, \ldots, x_n$, *t in number, are connected by the same linear relation of the form*

*Vol. XVII (1910), p. 137.

$$(1) \qquad l_1 x_{s+1} + \ldots + l_t x_n + h = 0,$$

in which the coefficients $l_1, \ldots, l_t$ *do not all vanish, is that the omission of the coefficients of these unknowns from the matrix of the system will reduce the rank of the matrix.*

2. Proof. That the condition is necessary is immediately evident, exactly as in the special case $t=1$, which Professor Miller considers. To show that the condition is also sufficient, we proceed by the method of induction under the assumption that the sufficiency of the condition holds for $t-1$ unknowns, we wish to show that it must also hold for t unknowns.

Let us suppose, therefore, that the omission of the coefficients of $x_{s+1}, \ldots, x_n$ from the matrix of the system causes a reduction in the rank of the matrix. If it happens that the omission of the coefficients of $t-1$ of these unknowns would alone suffice to reduce the rank of the matrix, then by the assumption just made these $t-1$ unknowns would be connected by a linear relation and the sufficiency of the condition would be proved.

Hence we may confine ourselves to the case in which a reduction of rank is not caused by the omission of the coefficients of $x_{s+2}, \ldots, x_n$, say, but only by the further omission of the coefficients of x_{s+1}. Let $r, \leqq s+1$, be the rank of the matrix of the system; then the partial matrix A_1 formed by the coefficients of $x_1, \ldots, x_s, x_{s+1}$ is also of rank r, while the matrix A' formed by the coefficients of $x_1, \ldots, x_s$ is of rank $r-1$. Consequently there must exist in A_1 at least one non-vanishing r-rowed determinant l_1 whose last column involves coefficients of x_{s+1}. There is no loss of generality in taking as the elements of this determinant the coefficients of $x_1, \ldots, x_{r-1}, x_{s+1}$ in the first r equations of the system. That is,

$$(2) \qquad l_1 = \Sigma \pm a_{11} \ldots a_{r-1,\, r-1}\, a_{r,\, s+1} \neq 0.$$

Let us now substitute for $x_{s+1}, \ldots, x_n$ in the equations of the system T any set of values that satisfy them; the resulting system T' in the s unknowns $x_1, \ldots, x_s$ will be consistent; and since the rank of its matrix A' is $r-1$, the rank of its augmented matrix

$$B' = \left\| \begin{array}{l} a_{11} \ldots a_{1s}\; a_{1,\, s+1} x_{s+1} + \ldots + a_{1n}\, x_n + k_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1} \ldots a_{ms}\; a_{m,\, s+1} x_{s+1} + \ldots + a_{mn}\, x_n + k_m \end{array} \right\|$$

is also $r-1$*; that is, every r-rowed determinant of B' must vanish. Consider the determinant $\triangle$ common to the first r rows and the first $r-1$ col-

*Cf. Capelli, *Rivista di Matematica*, Vol. 2 (1892), p. 54. English statements of Capelli's theorem are to be found in Bocher's *Introduction to Higher Algebra*, p. 46, Theorem 1, and in Miller's paper cited above.

umns together with the last column of B'. Since the elements of the last column of $\triangle$ are linear functions of $x_{s+1}, \ldots, x_n$, $\triangle$ itself can be expressed as a linear function of the same variables, and their coefficients are r-rowed determinants involving the elements of the augmented matrix of the original system T. Thus

$$\triangle \equiv l_1 x_{s+1} + \ldots + l_t x_n + h = 0;$$

and l_1, being precisely the determinant of equation (2), does not vanish. We have now shown that all the values of $x_{s+1}, \ldots, x_n$ that satisfy the equations of the system T are connected by the same linear relation of the required form; therefore the theorem is proved, provided we can establish it for the case $t=1$.

But it is evident that the method we have used for passing from $t-1$ to t is also applicable to the case $t=1$, and so affords an independent proof of Professor Miller's theorem, besides completing the proof of this.

3. A number of corollaries are immediately deducible. For instance, if the unknowns $x_{s+1}, \ldots, x_n$ are unrestricted except by a single linear relation, or in other words, if in the solutions of the system T the values of these unknowns are precisely those that satisfy one linear relation of the form (1), then the rank of the matrix will be reduced exactly one unit by the omission of the coefficients of these unknowns, and conversely. On the other hand, if there exist i (and only i, say) independent and compatible linear relations connecting the t unknowns $x_{s+1}, \ldots, x_n$, then the rank r of the matrix will be reduced exactly i units, and conversely; evidently $i \leqq t$ and $i \leqq r$; if $i=t$, each of the t unknowns is restricted to a single value and the rank is reduced one unit by the omission of the coefficients of each of the t unknowns.

Moreover, if the relation (1) actually involves all of the t unknowns $x_{s+1}, \ldots, x_n$, that is, if $l_i \neq 0$ $(i=1, \ldots, t)$, and if there is no other linear relation between them, then the omission of the coefficients of all of these unknowns, and not merely of some of them, will reduce the rank of the matrix, and the amount of the reduction is one unit. The converse again holds.

Similarly, if the t unknowns are connected by i independent and compatible linear relations that actually involve all the unknowns, then the reduction in rank due to the omission of the coefficients of all of these t unknowns is equal to i, while the reduction due to the omission of the coefficients of a part of the unknowns is less than i, and conversely.

Classification.

4. Let us now consider the bearing of this theorem and its corollaries on the classification of systems of linear equations. Let T be a consistent

system of rank r in n unknowns. After a suitable permutation of the unknowns among themselves, the complete solution of T may be written in the form

$$S \begin{cases} x_1 = l_{11}x_{r+1} + \ldots + l_{1,\,n-r}\,x_n + h_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_r = l_{r1}x_{r+1} + \ldots + l_{r,\,n-r}\,x_n + h_r, \end{cases}$$

which means that all the solutions of T are obtainable by giving all possible values to $x_{r+1}, \ldots, x_n$ and for each set of values of these unknowns finding the corresponding values of the other unknowns $x_1, \ldots, x_r$ from the equations S.

Each of the equations S is of the form of equation (1), since the coefficients of $x_1, \ldots, x_r$ cannot vanish. But the remaining coefficients l_{ij} may some or all of them vanish, and if we classify the systems T with respect to the vanishing or non-vanishing of these coefficients, that is, by determining which of the unknowns $x_{r+1}, \ldots, x_n$ are actually involved in the various equations of S, the different classes so obtained (see § 5) will be distinguished from one another and characterized by the particular sets of unknowns, the omission of whose coefficients from the matrix of T will cause reductions in the rank of the matrix.

Since no two of the equations S involve the same unknowns, those on the left being different in every case, therefore the reduction in rank caused by the omission of the coefficients of (a) all, or (b) a part of the unknowns actually involved in any one of the equations of S is equal to (a) one unit or (b) zero units.

We define the symbol $R_{x_\alpha \ldots x_\lambda}$ as the number of units of reduction in rank that is caused by the omission of the coefficients of $x_\alpha, \ldots, x_\lambda$ from the matrix of T; and we understand in every case by an equation of the form $R_{x_\alpha \ldots x_\lambda} = j$, that the omission of the coefficients of a part of these unknowns will reduce the rank less than j units, except when $j=0$. As an illustration, let T be a system of rank 3 in five unknowns whose complete solution S is of the form

$$\begin{cases} x_1 = l_{11}x_4 + l_{12}x_5 + h_1 \\ x_2 = l_{21}x_4 \qquad\quad\; + h_2 \\ x_3 = \qquad\qquad\qquad\; h_3 \end{cases}$$

where l_{11}, l_{12}, l_{21} are all different from zero; then the class of systems to which T belongs will be characterized by the equations $R_{x_1 x_4 x_5} = R_{x_2 x_4} = R_{x_3} = 1$; moreover,

$$R_{x_1 x_4} = R_{x_2} = R_{x_4} = 0, \quad R_{x_1 x_2 x_4 x_5} = R_{x_2 x_3 x_4} = 2, \quad \text{and} \quad R_{x_1 x_2 x_3 x_4 x_5} = 3.$$

5. Now the number of ways in which the $r(n-r)$ coefficients l_{ij} in the equations S can be chosen with respect to their vanishing or not is $2^{r(n-r)}$. But these do not all give essentially distinct classes of systems T. For if two such choices differ only in the order in which the unknowns are arranged, that is, if one can be obtained from the other by permuting either the $n-r$ unknowns $x_{r+1}, \ldots, x_n$ on the right or the r unknowns $x_1, \ldots, x_r$, on the left, then they give rise to essentially the same class; otherwise to distinct classes.

Let $N_{n,r}$ denote the number of classes of systems of consistent linear equations of rank r in n unknowns. Then $N_{n,r}$ is equal to the total number of combinations of $r(n-r)$ things arranged in the form of a rectangle with r rows and $n-r$ columns, provided that two combinations be regarded as identical, if one can be obtained from the other by permuting the rows of this rectangle or the columns or both. Obviously, $N_{n,r}=N_{n,n-r}$.

Query. Is there a convenient formula, old or new, giving the value of $N_{n,r}$ in terms of n and r?

For a few of the smaller values of n and r it is not difficult to find the corresponding values of $N_{n,r}$, as follows:

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	7	4	1		
$n=5$	1	5	13	13	5	1	
$n=6$	1	6	22	32	22	6	1

Geometrical Interpretation.

6. If we interpret $x_1, \ldots, x_n$ as the Cartesian coordinates of a point in a space S_n of n dimensions,* then a system T of consistent linear equations of rank r in these variables will stand for a system of linear (flat) spaces S_{n-1} intersecting in a linear space S_{n-r}; and the different classes we have been considering will correspond to the different degrees of parallelism† that can exist between S_{n-r} and the coordinate axes, coordinate planes, etc.

On the other hand, if the equations of T are homogeneous in the n variables, in which case they are always consistent, then another geometrical interpretation is possible. Namely, let the variables be the homogeneous coordinates of a point in a space S_{n-1}, referred to a coordinate simplex.‡ T will now stand for a system of linear spaces S_{n-2} interecting in a linear space

*Cf. Schoute, *Mehrdimensionale Geometrie*, Vol. 1, pp. 29, 125, and 137.
†Schoute, l. c., p. 34.
‡Schoute, l. c., p. 142.

S_{n-r-1}, except when $r=n$; in the latter case the spaces have no point in common. The classification we have been considering will correspond to the different degrees of intersection, including coincidence and non-intersection, that exist between S_{n-r-1} and the vertices, edges, faces, etc., of the coordinate simplex.

ILLUSTRATIONS.

7. In order to illustrate the general theory, a few of the simpler special cases will now be mentioned.

First, we put $n=3$ and enumerate the different classes of consistent systems T of linear non-homogeneous equations in the three variables x, y, z for each possible value of the rank r. Geometrically, T is a system of planes having at least one point in common, and referred to Cartesian axes, retangular or oblique, in ordinary space of three dimensions. Let S_{3-r} be the locus common to these planes.

$r=3$. Every system T has one solution. The number of classes of systems is $N_{3,3}=1$. This one class is characterized by the equations $R_x=R_y=R_z=1$, or by the single equation $R_{xyz}=3$. The common locus S_0 of the planes is a finite point.

$r=2$. T has ∞^1 solutions. The number of classes is $N_{3,2}=3$. Call these classes (2_1), (2_2), and (2_3), respectively.

(2_1) $R_{xz}=R_{yz}=1$. Moreover, $R_{xy}=1$.

The common locus S_1 is a (finite) line not parallel* to a coordinate plane.

(2_2) $R_x=R_{yz}=1$. S_1 is a line parallel to the plane $x=0$, but not parallel to a coordinate axis.

(2_3) $R_x=R_y=1$ or $R_{xy}=2$. S_1 is a line parallel to the z-axis.

$r=1$. T has ∞^2 solutions. The number of classes is $N_{3,1}=3$.

(1_1) $R_{xyz}=1$. S_2 is a plane not parallel to a coordinate axis.

(1_2) $R_{xy}=1$. S_2 is a plane parallel to the z-axis, but not parallel to a coordinate plane.

(1_3) $R_x=1$. S_2 is a plane parallel to $x=0$.

8. As a second illustration we take homogeneous equations and put $n=4$. Let x, y, z, w, be the homogeneous coordinates of a point in three-dimensional space referred to a coordinate tetrahedron. T is again a system of planes and S_{3-r} is their common locus, except when $r=4$.

$r=4$. One solution, $x=y=z=w=0$. $N_{4,4}=1$.

$R_x=R_y=R_z=R_w=1$. No common locus.

$r=3$. ∞^1 solutions. $N_{4,3}=4$.

(3_1) $R_{xw}=R_{yw}=R_{zw}=1$. S_0 is a (finite or infinite) point not lying in a coordinate plane.

*It is to be understood here that two parallel lines or planes may, in particular, be coincident lines or planes, and that a line parallel to a plane may be a line lying in the plane.

(3_2) $R_x=R_{yw}=R_{zw}=1$. S_0 is a point lying in the plane $x=0$, but not in a coordinate axis.

(3_3) $R_x=R_y=R_{zw}=1$. S_0 is a point in the line $x=y=0$, but not coinciding with a coordinate vertex.

(3_4) $R_x=R_y=R_z=1$. S_0 is the point $x=y=z=0$.

$r=2$. ∞^2 solutions. $N_{4,2}=7$.

(2_1) $R_{xzw}=R_{yzw}=1$. S_1 is a line not meeting a coordinate axis (either in a finite or an infinite point).

(2_2) $R_{xw}=R_{yzw}=1$. S_1 is a line meeting $x=w=0$, but not meeting any other coordinate axis.

(2_3) $R_{xw}=R_{yz}=1$. S_1 is a line meeting the two opposite coordinate axes $x=w=0$ and $y=z=0$, but not lying in a coordinate plane.

(2_4) $R_x=R_{yzw}=1$. S_1 is a line lying in the plane $x=0$, but not passing through a coordinate vertex.

(2_5) $R_{xz}=R_{yz}=1$. S_1 is a line passing through the point $x=y=z=0$, but not lying in a coordinate plane.

(2_6) $R_x=R_{yz}=1$. S_1 is a line passing through the point $x=y=z=0$, and lying in the plane $x=0$, but not coinciding with a coordinate axis.

(2_7) $R_x=R_y=1$. S_1 is the line $x=y=0$.

$r=1$. ∞^3 solutions. $N_{4,1}=4$.

(1_1) $R_{xyzw}=1$. S_2 is a plane not passing through a coordinate vertex.

(1_2) $R_{xyz}=1$. S_2 is a plane passing through the point $x=y=z=0$, but not containing a coordinate axis.

(1_3) $R_{xy}=1$. S_2 is a plane containing the line $x=y=0$, but not coinciding with a coordinate plane.

(1_4) $R_x=1$. S_2 is the plane $x=0$.

9. It may be worth while to add the four-dimensional interpretation of the seven classes (2_1), ..., (2_7) just given, for which $n=4$ and $r=2$, under the supposition that the equations of T are again non-homogeneous.

(2_1) S_2 is a plane having neither complete nor partial parallelism to any coordinate plane.

(2_2) S_2 is a plane half parallel to $x=w=0$, but neither completely parallel nor half parallel to any other coordinate plane.

(2_3) S_2 is a plane half parallel to the two opposite coordinate planes $x=w=0$ and $y=z=0$, but not (completely) parallel to a coordinate space S_3.

(2_4) S_2 is a plane parallel to the space $x=0$, but not parallel to a coordinate axis.

(2_5) S_2 is a plane parallel to the line $x=y=z=0$, but not parallel to a coordinate space.

(2_6) S_2 is a plane parallel both to the line $x=y=z=0$ and to the space $x=0$, but not completely parallel to a coordinate plane.

(2_7) S_2 is a plane completely parallel to $x=y=0$.

THE FOUNDER OF GROUP THEORY.

By G. A. MILLER, University of Illinois.

In the very useful *Encylcopédie des Sciences Mathematiques*, tome I, volume 1 (1909), page 432, it is stated that Cauchy may be considered as the founder of the theory of substitution groups. On the contrary, Easton affirms in his *Construtcive development of group theory*, 1902, page 43, "The proper founder of group theory is Evariste Galois." Again, Pierpont has stated in the *Bulletin of the American Mathematical Society*, vol. I (1895), page 196, "in the present brief note I cannot vindicate Lagrange's right to the title of creator of the theory of substitutions; but I hope by presenting a few examples of his methods to show the importance of considering him from this point of view." A still different view is expressed by Burkhardt in his important article entitled "Die Anfaenge der Gruppen theorie und Paolo Ruffini" published in the *Abhandlungen zur Geschichte der Mathematik*, 1892, pages 119-159. In concluding this paper he says: "It will probably not be possible to determine whether the essentials of the work of his friend Abbati, which includes the first complete proof of two fundamental theorems, are due to the inspiration of Ruffini, or whether we should regard this almost forgotten man alongside with Ruffini as one of the founders of group theory."

From the preceding paragraph it is clear that within the last twenty years four different names (Cauchy, Galois, Lagrange, and Ruffini) have been given by good authorities as those of founders of the theory of groups. Among others for whom this honor has been claimed Abel is perhaps the best known. In his well known *Synopsis der hoeheren Mathematik*, 1891, page 287, Hagen says "The theory of substitution groups was founded by Abel and Cauchy, and was developed to a certain extent principally by Galois and Jordan, especially with a view to the solution of algebraic equations." On the other hand, in Maillet's, Paris, thesis, 1892, we read "The founders of the theory (of substitution groups) at least in its present form are Galois and Cauchy." These quotations may suffice to show how difficult it is to determine the founder of a large subject even of comparatively recent origin. As regards the older subjects, for instance, analytic geometry and calculus, the difficulty is generally much greater.

We are inclined to attribute the honor of starting a given big theory to an individual just as we are prone to ascribe fundamental theorems to particular men, who frequently have added only a small element to the development of the theorem. Hence the statement that a given individual founded a big theory should not generally be taken very seriously. It adds, however, a pleasant human flavor and awakens in us a noble sense of admiration and appreciation. It is also of value in giving a historical setting and brings into play a sense of the dynamic forces which have contributed to its

development instead of presenting to us a cold static scene. Observations become more inspiring when they are permeated with a sense of development.

The earliest of the names proposed as founder of the group theory is that of Lagrange, and his claim is based upon his admirable paper on the algebraic solutions of equations, published in the Berlin *Nouveaux Mémoires*, 1770 and 1771. If we accept Lagrange as the founder of group theory this subject is now about one hundred forty years old. At any rate, all the later men for whom the honor of having founded group theory is claimed were inspired by the beautiful theorems of this memoir. Among these is the fundamental theorem that the order of a subgroup is a divisor of the order of the group, which is sometimes called Lagrange's theorem.* It is scarcely necessary to add that Lagrange did not state this theorem, article 104 of his memoir, in the form in which we give it now. His language seems however practically equivalent to the given statement and is more accurate than one would naturally infer from the incorrect statement in Pierpont's review in the *Bulletin of the American Mathematical Society*, volume I (1895), top of page 198.

From the given quotations it may be inferred that Cauchy is more commonly regarded as the founder of group theory than any of the others. The opening sentence of the preface of Burnside's work on this subject, "The theory of groups of finite order may be said to date from the time of Cauchy" is in accord with such an inference. We proceed to consider some reasons for this conclusion. In the first place, it is desirable to emphasize the fact that Cauchy's contributions to group theory may be conveniently divided into two parts which are separated by a period of about thirty years. The first of these consists almost entirely of two articles published in 1815 in volume 10 of the oldest extant mathematical periodical, *Journal de l'école polytechnique*, while the second part begins with volume 3 of his *exercises d'analyse et de physique mathematique*, 1844, and closes with the numerous articles which he published in the Paris *Comptes Rendus* during 1845 and 1846.

The first of these periods is subsequent to the works of Lagrange and Ruffini but it antecedes those of Abel and Galois, while the second period is subsequent also to the works of the latter. During this second period Cauchy made his most important as well as his most extensive contributions to our subject, and these are the contributions which appear to justify the claim that he is the founder of group theory. His contributions of 1815 are scarcely as meritorious as the earlier ones by Lagrange or Ruffini, and hence these would not justify the claim that he is the founder of this theory. The assumption that Cauchy is the founder of group theory therefore implies that this theory is less than seventy years old.

One of the fundamental theorems proved by Cauchy during the

*Cf. Pincherli, *Lezioni di Algebra Complementare*, 1909, p. 44.

second period of his activity along this line is, Every group whose order is divisible by a given prime (p) must have a subgroup of order p. This is sometimes called Cauchy's theorem, and it constitutes the most difficult element in the development of Sylow's theorem. After Lagrange proved that the order of a subgroup is a divisor of the order of the group, and Ruffini established the theorem that it is not always possible to find a subgroup whose order is a given divisor of the order of the group (Ruffini's theorem), it was of great interest to establish the fact that a subgroup exists for every possible prime divisor. These three theorems, associated with the names of Lagrange, Ruffini, and Cauchy, respectively, constitute the most indispensable elements for the further development of the subject.

While Cauchy's theorem is comparable with those of Lagrange and Ruffini as regards fundamental importance for the further development of group theory, its proof demands a decidedly deeper insight into the nature and structure of a group. In this respect a fundamental theorem proved by Cauchy during his first period of group-theoretic activity is more nearly comparable with those of Lagrange and Ruffini. This theorem established the fact that the symmetric group of degree n cannot involve a subgroup whose index lies between 2 and p, where p is the largest prime which divides n. This theorem is a second extension of one due to Ruffini (1799), the first extension having been made by Abbati in 1803. It was extended still further by Bertrand, Serret, and others, and is of great importance in the study of the number of different values which a function may assume when its variables are permuted in every possible way.

Having considered the most important theorems contributed by Lagrange, Ruffini, and Cauchy towards the development of group theory it is of interest to inquire into the contributions of Galois, whose name has also been advanced as that of an individual founder of this subject. Perhaps his most important direct contribution is the introduction of the concept of modulus into group theory, a concept which Gauss had introduced into number theory about thirty years earlier (1801). In group theory, the modulus is generally known as invariant subgroup, although Jordan used the term modulus, at least indirectly, for this concept as early as 1873, in the *Bulletin of the French Mathematical Society*, volume I, page 46. In group theory this important concept is known by the following names, in chronological order: proper divisor, modulus, distinguished subgroup, invariant subgroup, monotypic subgroup, self-conjugate subgroup, normal divisor, and autojugan divisor. The multitude of names is perhaps partly due to the many different ways of approach to this concept.

Although Cauchy has contributed a number of important theorems to the development of group theory his claims as founder of this theory are more strongly supported by the fact that during the second period of his activity along this line he made the first systematic study of the theory of substitution groups, under the name of systems of conjugate substitutions.

While his developments are often prolix and involve some inaccuracies, they have placed a considerable part of the theory of substitutions into an easily accessible form and have been a source of inspiration for many of his successors. As an instance of an inaccuracy we may cite his statement (on page 443, volume 9, of the first series of his works) that a primitive group whose degree is a prime number increased by one cannot be simply transitive. It is very easily seen that the symmetric group of degree 9 can be represented as a simply transitive primitive group of degree $84=83+1$. His enumeration of the possible orders of groups of degree 6, on page 493 of the same volume, is also far from correct; but this should perhaps not surprise us in view of the large number of errors in the published enumerations of possible substitution groups.

The preceding considerations neither prove nor disprove the justice of the claim that Cauchy is the founder of group theory even if they tend to support this view. It has been our aim to exhibit a few of the elements involved in such a question, and especially to point out that many efficient workers are needed for the development of a great subject. Mathematical subjects gain in attractiveness if we can associate with them an intelligent insight into their growth and a due appreciation of the costly heritage involved in their fundamental theorems. To this end it is desirable to associate one or more founders with each of the modern subjects.

PERFECT NUMBERS.

By T. M. PUTNAM, University of California.

The theory of numbers, probably more than any other branch of mathematics, offers problems that are very easy to state and formulate completely, but extremely difficult to solve. There are many that have baffled even trained workers in this field, who have been obliged to content themselves in many cases with but partial resolutions of the questions. These very often appear as isolated, artificial problems whose solution would apparently add very little to the main body of theory. But sometimes there is an historical interest attached, which coupled with an alluring simplicity of formulation attracts investigators toward it. There is always the possibility, too, that the pursuit of solutions of even these elusive problems may lead to the discovery of mathematical relations, or processes that are new and of much more general application than to the immediate problem to be solved.

Some such justification may be necessary for research concerning the existence or relations of perfect numbers. Indeed, Fermat was led by this problem to some of his most important theorems. It is moreover a problem of much historic interest.

A perfect number is defined to be one that is equal to the sum of all its divisors exclusive of itself. If one denotes such a number by m, and the sum of all its factors including itself by $\sigma(m)$, then the defining relation is

$$\sigma(m)=2m.$$

If $m=a^\alpha b^\beta \dots l^\gamma$ where a, b, ..., l are distinct primes, then

$$\sigma(m)=(1+a+a^2+\dots+a^\alpha)(1+b+b^2+\dots+b^\beta)\dots(1+l+l^2+\dots+l^\gamma).$$

Hence the defining condition becomes

$$2\,a^\alpha b^\beta \dots l^\gamma = (1+a+\dots+a^\alpha)\dots(1+l+\dots+l^\gamma).$$

Euclid showed that all numbers of the form $2^{n-1}(2^n-1)$, where 2^n-1 is a prime, are perfect numbers; but it was Euler who first showed that all even perfect numbers are necessarily of this form. These theorems are easily verified by the above defining condition.

The smallest perfect numbers given by Euclid's formula are 6 and 8, but it has so far produced only nine such numbers, owing to the difficulty of determining whether or not 2^n-1 is a prime, when n is large. The known cases are for the values $n=2$, 3, 5, 7, 13, 17, 19, 31, and 61. It is not known whether or not, this formula contains an infinite number of perfect numbers.

The theory of *even* perfect numbers is therefore fairly complete. But no *odd* perfect number has been found, nor has their existence either been proved or disproved. It is possible, however, to set up certain restrictive theorems concerning them, assuming that they do exist, which so hedge them about that the chances of any surviving are at least extremely small.

Some of the more important of these theorems will be stated below, and one new one (v. 7) will be given with proof and some applications.

1. If an odd perfect number exists it has the form $p^k A^2$, where p and k are both of the form $4n+1$, and p is a prime not dividing A. (Lucas, *Theorie des Nombres*, p. 425).

2. From (1) follows that there are no perfect numbers of the form $4h+3$.

3. Bourlet gives the following (*Nouv. Ann.*, 1896, p. 297): If the divisors of a perfect number be denoted by d_i then $\Sigma\frac{1}{d_i}=2$.

4. It follows from (3) that no divisor of a perfect number can be perfect and there must be more divisors than the least prime.

5. There are no perfect numbers of the forms $p^k a^2b^2\dots l^2$ and $p^k a^4b^4\dots l^4$, where p, a, b, ..., l are primes.

6. $\frac{m}{\sigma(m)} > (1-\frac{1}{a})(1-\frac{1}{b})\ldots(1-\frac{1}{l}) < 2$ (Bourlet, l. c.) From this follows that $\phi(m) < \frac{1}{2}m$, *i. e.*, less than half the numbers smaller than m are prime to it. This theorem and the next one derived from it are useful in ruling out certain types of odd numbers from consideration.

7. If we denote by r the number of distinct primes in a number m, and suppose that all these primes are greater than $\frac{r}{\log 2}+1$, the number m cannot be perfect.

To prove this take the relation given in (6):

$$\frac{m}{\sigma(m)} > \frac{1}{1+\frac{1}{a-1}} \cdot \frac{1}{1+\frac{1}{b-1}} \cdots\cdots \frac{1}{1+\frac{1}{l-1}}.$$

But by hypothesis each prime is greater than

$$\frac{r}{\log 2}+1, \text{ hence } \frac{1}{1+\frac{1}{a-1}} > \frac{1}{1+\frac{\log 2}{r}}, \text{ etc.}$$

Therefore, $\frac{m}{\sigma(m)} > \left(\frac{1}{1+\frac{\log 2}{r}}\right)^r > \left(\frac{1}{1+\frac{\log 2}{n}}\right)^n \quad (n>r)$.

Hence, $\frac{m}{\sigma(m)} > \lim_{n=\infty}\left(\frac{1}{1+\frac{\log 2}{n}}\right)^n$, that is, $\frac{m}{\sigma(m)} > \frac{1}{e^{\log 2}}$, or $2m > \sigma(m)$.

Hence m cannot be perfect.

Since $\frac{1}{\log 2} = 1.45+$ one can state the theorem for working purposes in the form: A number with r distinct prime factors all of which are greater than $\frac{3}{2}r+1$ cannot be perfect.

Applications of (7).

a) There are no odd perfect numbers of the form a^α. Here $j=1$, $\frac{3}{2}j+1=\frac{5}{2}$, and since 3 is the smallest possible prime, the proof follows at once.

b) There are no odd perfect numbers with only two distinct prime factors. Here $\frac{3}{2}j+1=4$. The only possible type is then $3^\alpha \cdot p^\beta$. But

$\frac{m}{\sigma(m)} > (1-\frac{1}{3})(1-\frac{1}{p})$, or $\frac{m}{\sigma(m)} > \frac{2}{3}(1-\frac{1}{p})$. But $\frac{m}{\sigma(m)} = \frac{1}{2}$, if m is perfect. Hence,

$$\tfrac{1}{2} > \tfrac{2}{3}(1-\frac{1}{p}), \text{ or } p<4.$$

Hence there are no possible solutions.

c) If an odd perfect number could exist with three distinct primes, then one of these would have to be 3 or 5; for, $\frac{3}{2}j+1=5\frac{1}{2}$. But if we suppose that the form is $m=3^{\alpha} \cdot p^{\beta} \cdot q^{\alpha}$, then since

$$(1-\tfrac{1}{3})(1-\frac{1}{p})(1-\frac{1}{q}) < \tfrac{1}{2}, \quad (1-\frac{1}{p})(1-\frac{1}{q}) < \tfrac{3}{4}.$$

If $p=7$ and $q=11$, then $(1-\frac{1}{p})(1-\frac{1}{q}) > \frac{3}{4}$, and larger values of p and q would also be impossible in this inequality. It follows therefore that one of them must be equal to 5. The form is then $m=3^{\alpha} \cdot 5^{\beta} \cdot p^{\gamma} \ldots$ The same form results if we suppose 5 present instead of 3, initially.

Since $1-\frac{1}{p} < \frac{15}{16}$ it follows that $p<16$.

The possible cases are then for $p=7$, 11, 13. Sylvester has shown that none of these combinations of three primes can lead to perfect numbers (*Comptes Rendus*, 1888). Hence no odd perfect number exists with less than four primes. Bourlet, investigating possible perfect numbers with four primes shows that none can exist less than 2,197,845.

This last theorem shows that if the prime factors of a number are all sufficiently large there is no possibility of it being perfect, no matter what the exponents may be. On the other hand, the third theorem expresses a relation involving all the divisors which implicitly involves the exponents of the prime factors of the number in such a way that they cannot exceed certain upper limits.

PROBLEM. Show that if an odd perfect number exists, say $p^k A^2$, then A has at least one prime factor smaller than p.

NOTE.—It may be of interest to some of our readers to know that Professor Benjamin Peirce, in 1832, *Mathematical Diary*, page 267, showed that there is no perfect number of the form a^m, $a^m b^n$, $a^m b^n c^p$, a, b, c being prime numbers greater than unity. ED. F.

PROBLEMS FOR SOLUTION.

ALGEBRA.

337. **Proposed by I. M. CURTISS, Brooklyn, N. Y.**

Three regiments move north as follows: B is 20 miles east of A; C is 20 miles south of B, and each marches 20 miles between the hours of 5 a. m. and 3 p. m. A horseman with a message from C starts at 5 a. m. and rides north till he overtakes B, then sets a straight course for the point at which he calculates to overtake A, then sets a straight course for the next point at which he will again overtake B, then rides south to the point where he first overtook B, reaching that point at the same time as C, namely 3 p. m. What uniform rate of travel enabled the messenger to do this?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let x=time required to overtake B. He travels $20+2x$ miles. Hence $\frac{20+2x}{x}$=his rate. Let y=time to go from B to A. He travels $2\sqrt{(100+y^2)}$ miles. $\frac{2\sqrt{(100+y^2)}}{y}$=his rate. After reaching B a second time he has left $10-x-2y$ hours to go $2x+4y$ miles.

$\therefore \frac{2x+4y}{10-x-2y}$=his rate. But his rate is uniform. Hence we get

$$\frac{20+2x}{x}=\frac{2\sqrt{(100+y^2)}}{y}, \text{ or } 5x^2=5y^2+xy \ldots (1).$$

$$\frac{20+2x}{x}=\frac{2x+4y}{10-x-2y}, \text{ or } x^2+2xy=50-10y \ldots (2).$$

If $x=vy$ in (1), we get $5v^2-v=5$ or $v=1.10499$.
$\therefore x=1.10499y$. This in (2) gives $14.41996y^2+10y=50$ or $y=1.54737$.
$x=1.70983$. Rate$=20/x+2=13.69707$ miles an hour.

Also solved by V. M. Spunar. A. H. Holmes, and J. Scheffer.

338. **Proposed by R. D. CARMICHAEL, Princeton University.**

Prove that $\pi=3+\frac{1}{3}\cdot\frac{1}{1.2}-\frac{1}{5}\cdot\frac{1}{2.3}+\frac{1}{7}\cdot\frac{1}{3.4}-\frac{1}{9}\cdot\frac{1}{4.5}+\ldots$

Solution by S. LEFSEHETZ, East Pittsburg, Pa.

Let $S(x)=\sum_1^\infty (-1)^{n+1}\frac{x^{2n}}{n(n+1)(2n+1)}=\sum_1^\infty\left[\frac{1}{n}+\frac{1}{n+1}-\frac{4}{2n+1}\right](-1)^{n+1}x^{2n}$

$$=\sum_1^\infty (-1)^{n+1}\frac{x^{2n}}{n}+\sum_1^\infty (-1)^{n+1}\frac{x^{2n}}{n+1}+4\sum_1^\infty(-1)^2\frac{x^{2n}}{2n+1}$$

$$=L(1+x^2)-\frac{1}{x^2}L\,(1+x^2)+1+\frac{4}{x}(\tan^{-1}x-x).$$

$$\therefore S(+1)=4(\frac{\pi}{4}-1)+1=\frac{1}{3.1.2}-\frac{1}{5.2.3}+\ldots$$

$$\therefore \pi=3+\frac{1}{3}.\frac{1}{1.2}-\frac{1}{5}.\frac{1}{2.3}+\ldots$$

Also solved by G. B. M. Zerr, and V. M. Spunar.

339. **Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.**

Prove that if $a_1 \nless 2$ and $a_n=a_{n-1}{}^2-2$, $\frac{1}{a_1}+\frac{1}{a_1\,a_2}+\frac{1}{a_1\,a_2\,a_3}+\ldots$ $=\frac{1}{2}[a_1-\sqrt{(a_1^2-4)}]$.

Solution by the PROPOSER.

Let the roots of the equation $x^2-a_1x+1=0$ be α and $1/\alpha$. Then

$$a_1=\alpha+\frac{1}{\alpha},$$

$$a_2=\alpha^2+\frac{1}{\alpha^2},$$

$$a_3=\alpha^4+\frac{1}{\alpha^4},$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

The series becomes

$$\frac{1}{\alpha+\frac{1}{\alpha}}+\frac{1}{\left(\alpha+\frac{1}{\alpha}\right)\left(\alpha^2+\frac{1}{\alpha^2}\right)}+\ldots+\left(\alpha-\frac{1}{\alpha}\right)\left[\frac{1}{\alpha^2-\frac{1}{\alpha^2}}+\frac{1}{\alpha^4-\frac{1}{\alpha^4}}+\frac{1}{\alpha^8-\frac{1}{\alpha^8}}\ldots\right]$$

Since $\frac{1}{\alpha^2-\frac{1}{\alpha^2}}=\frac{1}{2}\left(\frac{\alpha+\frac{1}{\alpha}}{\alpha-\frac{1}{\alpha}}-\frac{\alpha^2+\frac{1}{\alpha^2}}{\alpha^2-\frac{1}{\alpha^2}}\right)$,

each term of the series may be written as the difference of two fractions, *i. e.*

$$\frac{\alpha-\frac{1}{\alpha}}{2}\left[\left(\frac{\alpha+\frac{1}{\alpha}}{\alpha-\frac{1}{\alpha}}-\frac{\alpha^2+\frac{1}{\alpha^2}}{\alpha^2-\frac{1}{\alpha^2}}\right)+\left(\frac{\alpha^2+\frac{1}{\alpha^2}}{\alpha^2-\frac{1}{\alpha^2}}-\frac{\alpha^4+\frac{1}{\alpha^4}}{\alpha^4-\frac{1}{\alpha^4}}\right)+\ldots.\right]$$

$$=\frac{\alpha-\frac{1}{\alpha}}{2}\left[\frac{\alpha+\frac{1}{\alpha}}{\alpha-\frac{1}{\alpha}}-\frac{\alpha^{2^n}+\alpha^{-2^n}}{\alpha^{2^n}-\alpha^{-2^n}}\right].$$

When $\alpha>1$, the limit of this sum is

$$\frac{\alpha-\frac{1}{\alpha}}{2}\left(\frac{\alpha+\frac{1}{\alpha}}{\alpha-\frac{1}{\alpha}}-1\right)=\frac{1}{\alpha}.$$

Therefore, the sum of the series is the smaller root of the equation $x^2-a_1x+1=0$, viz., $\frac{a_1-\sqrt{(a_1^2-4)}}{2}$.

NOTE.—This furnishes a very rapid method for finding square roots to a considerable number of decimals. Example.—Let $a=16$,

$$8-3\sqrt{7}=\frac{1}{16}+\frac{1}{16.254}+\frac{1}{16.254.645.64514}+\ldots$$

and three terms of the series give $\sqrt{7}$ to 18 decimals, four terms will give 37 decimals, etc.

Also solved by S. Lefsehetz.

GEOMETRY.

365. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Given the coordinates of the four vertices of the tetrahedron, (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3); (x_4, y_4, z_4): find volume and express it by a determinant.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and I. W. SMITH, A. M., Assistant Professor of Mathematics, North Dakota Agricultural College.

Let $\triangle$ =area of triangular base BCD of the tetrahedron, p the perpendicular from the vertex A on the base BCD, V its volume, $(x-x_2)\cos\alpha+(y-y_2)\cos\beta+(z-z_2)\cos\gamma=0$ the equation to the plane of BCD.

Then $\triangle\cos\gamma$ is the projection of the area BCD on the plane xy, and (x_2, y_2), (x_3, y_3), (x_4, y_4) are its angular points.

$$\therefore 2\triangle\cos\gamma=\begin{vmatrix}1, & 1, & 1\\ x_2, & x_3, & x_4\\ y_2, & y_3, & y_4\end{vmatrix}=\begin{vmatrix}x_3-x_2, & x_4-x_2\\ y_3-y_2, & y_4-y_2\end{vmatrix}.$$

Similarly,

$$2\triangle\cos\beta=\begin{vmatrix}z_3-z_2, & z_4-z_2\\ x_3-x_2, & x_4-x_2\end{vmatrix},\quad 2\triangle\cos\alpha=\begin{vmatrix}y_3-y_2, & y_4-y_2\\ z_3-z_2, & z_4-z_2\end{vmatrix}.$$

Also, $-p=(x_1-x_2)\cos\alpha+(y_1-y_2)\cos\beta+(z_1-z_2)\cos\gamma$.

$$\therefore -6V=-2\triangle p=2\triangle\cos\alpha(x_1-x_2)+2\triangle\cos\beta(y_1-y_2)+2\triangle\cos\gamma(z_1-z_2)$$

$$=(x_1-x_2)\begin{vmatrix}y_3-y_2, & y_4-y_2\\ z_3-z_2, & z_4-z_2\end{vmatrix}+(y_1-y_2)\begin{vmatrix}z_3-z_2, & z_4-z_2\\ x_3-x_2, & x_4-x_2\end{vmatrix}$$

$$+(z_1-z_2)\begin{vmatrix}x_3-x_2, & x_4-x_2\\ y_3-y_2, & y_4-y_2\end{vmatrix}$$

$$=\begin{vmatrix}x_1-x_2, & x_3-x_2, & x_4-x_2\\ y_1-y_2, & y_3-y_2, & y_4-y_2\\ z_1-z_2, & z_3-z_2, & z_4-z_2\end{vmatrix}=-\begin{vmatrix}1, & 1, & 1, & 1\\ x_1-x_2, & 0, & x_3-x_2, & x_4-x_2\\ y_1-y_2, & 0, & y_3-y_2, & y_4-y_2\\ z_1-z_2, & 0, & z_3-z_2, & z_4-z_2\end{vmatrix}$$

Multiply the first row of this last determinant by x_2, y_2, z_2 and add to the second, third, fourth, respectively, we get

$$V=\tfrac{1}{6}\begin{vmatrix}1, & 1, & 1, & 1\\ x_1, & x_2, & x_3, & x_4\\ y_1, & y_2, & y_3, & y_4\\ z_1, & z_2, & z_3, & z_4\end{vmatrix}$$

Also solved by S. G. Barton, and V. M. Spunar.

366. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Astronomy, Manchester, N. H.

Construct a triangle, having given the base, vertical angle, and difference of altitude and difference of other two sides.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Mo.

Let A be the given vertical angle, a the given side, and d=given difference. Then $h-(b-c)=d$, or $b-c=h-d$...(1).

We have also $b^2+c^2-2bc\cos A=a^2$, which by means of (1) changes into $4bc\sin^2\frac{1}{2}A=a^2-(h-d)^2$, whence $bc=\dfrac{a^2-(h-d)^2}{4\sin^2\frac{1}{2}a}$; but $bc\sin A=ah$.

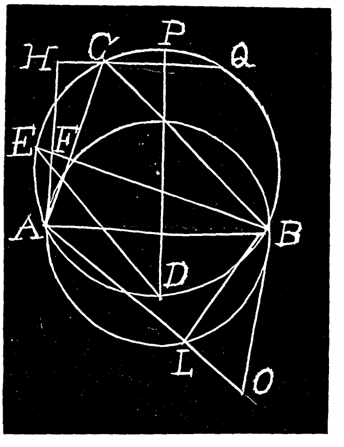
$\therefore\ \dfrac{a^2-(h-d)^2}{4\sin^2\frac{1}{2}a}=\dfrac{an}{\sin A}$, and thus we finally get the quadratic equa-

tion in h, $h^2-2(d-a\tan\frac{1}{2}A)h=a^2-d^2$. This quadratic is easily constructed, and after having h the construction of the triangles is very easy.

Solved similarly by V. M. Spunar.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $AB=a$ be the base, ACB the given vertical angle, $p=z-(q-y)$ where z=altitude; x, y the remaining sides, respectively. On AB describe the segment of the circle containing the given vertical angle. Draw the diameter PD perpendicular to AB. Take $DE=AB$ and draw BE. Erect AF perpendicular to AB and also describe a circle on AB as diameter. Draw a line $AL=p$ meeting this circumference in L. Draw LB. Take $LO=AL-AF$ where LO is AL produced. Draw BO. Produce AF to H making $AH=BO+LO$, and draw HCQ parallel to AB. Then ACB is the triangle required. For $xy\sin C=az$, $x-y=z-p$, $a^2=x^2+y^2-2xy\cos C$. Hence $(z-p)^2=a^2-2xy(1-\cos C)=a^2-2az\tan\frac{1}{2}C$.



$\therefore z=p-a\tan\frac{1}{2}C+\sqrt{[(p-a\tan\frac{1}{2}C)^2+a^2-p^2]}$.

$\angle ABE=\frac{1}{2}C$, $AF=a\tan\frac{1}{2}C$, $AL=p$, $BL=\sqrt{[a^2-p^2]}$.
$LO=p-a\tan\frac{1}{2}C$, $BO=\sqrt{[(p-a\tan\frac{1}{2}C)^2=a^2-p^2]}$.
$\therefore AH=z$ and ABC is the required triangle.

CALCULUS.

292. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the partial differential equation, $x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}=axy$.

Solution by PROF. W. W. BEMAN, University of Michigan.

This is problem 12, page 297, of Johnson's *Differential Equations*.

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{dxy}.$$

From the first two fractions,

$$\frac{1}{x}-\frac{1}{y}=i. \quad \therefore x=\frac{y}{1+cy}.$$

Substituting this value of x in the last two fractions,

$$\frac{dz}{a}=\frac{dy}{1+cy}, \text{ or } z=\frac{a}{c}\log(1+cy)+c'.$$

Replacing c by $\frac{1}{x}-\frac{1}{y}$, $z=\frac{axy}{y-x}\log\frac{y}{x}+\phi\left(\frac{1}{x}-\frac{1}{y}\right)$.

Also solved similarly by J. Scheffer, and G. B. M. Zerr.

293. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Find the length of the integral curve of the differential equation $(y^2x^{\frac{4}{3}}+2)dx-x^{\frac{4}{3}}dy=0$ between $x_1=1$ and $x_2=8$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $y=-1/z$, $x^{-\frac{1}{3}}=v$, then the equation becomes $dz+6z^2dv+3v^{-4}dv=0$.

Now, let $z=\frac{1}{6v}+\frac{u}{v^2}$.

Then $v^2du+6u^2dv+3dv=0$, or $\frac{du}{3+6u^2}=-\frac{dv}{v^2}$.

$$\therefore \frac{1}{3\sqrt{2}}\tan^{-1}(u\sqrt{2})=\frac{1}{v}+a=a+x^{\frac{1}{3}}\ldots(1).$$

$$\therefore u=\frac{1}{\sqrt{2}}\tan[3\sqrt{2}(a+x^{\frac{1}{3}})]\ldots(2).$$

$$\therefore y=-\frac{6\sqrt{2}}{6x^{\frac{2}{3}}\tan[3\sqrt{2}(a+x^{\frac{1}{3}})]+x^{\frac{1}{3}}\sqrt{2}} \text{ is the equation.}$$

From (1), $u=\frac{1}{\sqrt{2}}\tan\left[\frac{3\sqrt{2}(av+1)}{v}\right]$.

$$S=\int\sqrt{[1+(du/dv)^2]}dv=\int_{\frac{1}{2}}^{1}\frac{1}{v^2}\sqrt{v^4+9\sec^4\left[\frac{3\sqrt{2}(av+1)}{v}\right]}dv.$$

294. Proposed by C. N. SCHMALL, New York City.

Examine the function, $f(x)=\frac{(x-1)(x-2)}{(x-3)}$ and determine why its *minimum* value is *greater* than its maximum.

Solution by PROF. F. L. GRIFFIN, Ph. D., Williams College.

The derivative, $f'(x)=\dfrac{x^2-6x+7}{(x-3)^2}$, vanishes for $x=3\pm\sqrt{2}$, changing its sign from + to − for the smaller of these values and from − to + at the larger. Thus the function decreases from its maximum value $f(3-\sqrt{2})$ to its minimum $f(3+\sqrt{2})$, *except that it passes through* ∞ at $x=3$, and thus it is possible for the minimum to exceed the maximum. The same fact holds for the more general function of the same type: $F(x)=\dfrac{(x-a)(x-b)}{x-c}$, if $c>a$ and $c>b$; but the maximum of $F(x)$ exceeds the minimum if $c<a$ and $c<b$. Thus the property proposed for explanation depends not so much on the mere numerical values, nor even on the fact of an infinite discontinuity, as upon the order of the zeros of the numerator and denominator.

Also solved by S. G. Barton, J. E. Sanders, V. M. Spunar, and J. Scheffer.

MECHANICS.

245. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A body moves with constant speed in the circumference of an ellipse. Find the rate of approach (1) to the center, (2) to one of the foci, for any point in the ellipse.

Solution by PROF. F. L. GRIFFIN, Ph. D., Williams College.

Differentiating $b^2x^2+a^2y^2=a^2b^2$ with respect to the time, we have $\dfrac{dx}{dt}/a^2y=-\dfrac{dy}{dt}/b^2x,\ =-\lambda$ (say), whence if k denote the constant speed, $k^2=\left(\dfrac{ds}{dt}\right)^2=\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2=\lambda^2[a^4y^2+b^4x^2]$, or $\lambda=k/\sqrt{[a^4y^2+b^4x^2]}$. The negative sign should be taken if the motion is clockwise.

Hence, $\dfrac{dx}{dt}=-ka^2y/\sqrt{[a^4y^2+b^4x^2]}$ and $\dfrac{dy}{dt}=kb^2x/\sqrt{[a^4y^2+b^4x^2]}$.

(I) Now the radius vector from the center to (x, y) is given by $r^2=x^2+y^2$; whence the rate of approach to the center is

$$-\frac{dr}{dt}=-\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)/r=-\frac{kxy(b^2-a^2)}{r\sqrt{[a^4y^2+b^4x^2]}}=\frac{ka^2e^2xy}{\sqrt{[(a^4y^2+b^4x^2)(x^2+y^2)]}}.$$

(II) The radius vector from the left-hand focus $(-ae, 0)$ to (x, y) is given by: $r^2=(x+ae)^2+y^2$; whence

$$-\frac{dr}{dt}=-\left[(x+ae)\frac{dx}{dt}+y\frac{dy}{dt}\right]/r=\frac{ka^2ey(ex+a)}{\sqrt{[a^4y^2+b^4x^2)(x^2+y^2)]}}.$$

REMARK. In (I) the sign changes with either x or y, that is, at the ends of either axis; while in (II) the sign changes only at the ends of the major axis. The factor $ex+a$ vanishes only on the left-hand directrix; hence not in the ellipse.

Also solved by G. B. M. Zerr, and J. Scheffer.

246. Proposed by A. M. HARDING, Adjunct Professor, University of Arkansas, Fayetteville, Ark.

A pentagon $ABCDE$, formed of equal uniform heavy rods connected by smooth joints at their ends, is supported symmetrically in a vertical plane with A uppermost, and AB and AE in contact with two smooth pegs in the same horizontal line. Prove that if the pentagon is regular, the pegs must divide AB and AE each in the ratio $1+\sin(\pi/10):3\sin(\pi/10)$. Jeans' *Theoretical Mechanics*, page 112, number 13.

Solution by PROF. F. L. GRIFFIN, Ph. D., Williams College.

Denote by P the point of contact of AB with one peg, and let $AP=2m$ and $PB=2n$. Also let the weight of each rod by $2W$, and the resistance exerted by each peg be R.

Consider first the rod AB, making an angle of 36° with the horizontal. At A the reaction of AE is horizontal (say X_a); for if there were a vertical component, then AB would exert upon AE an oppositely directed vertical component, contrary to the hypothesis of symmetry. At B the reaction of BC is unknown; denote its components by X_b and Y_b. The other forces applied to AB are its weight $2W$, and the reaction R inclined 54° to the horizontal. Equating to zero the sums of the vertical components, horizontal components, and moments about P we obtain (since the distance from P to the midpoint of AB is $2m-(m+n)=m-n$).

(1), (2) $\qquad Y_b+R\sin54^\circ-2W=0, \quad X_a+X_b+R\cos54^\circ=0,$

(3) $\qquad X_b 2n\cos54^\circ+Y_b 2n\sin54^\circ+2W(m-n)\sin54^\circ-X_a 2m\cos54^\circ=0.$

Now R is easily found by considering the pentagon as a whole, to which are applied three external forces, R, R and the weight $10W$. The vanishing of the vertical component gives

(4) $\qquad 2R\sin54^\circ-10W=0, \quad R=5W\csc54^\circ.$

To obtain X_b consider the rod BC, to which is applied at B the reaction of AB whose components are respectively $(-X_b)$ and $(-Y_b)$. Equating to zero the resultant moment about C, we have, since $BC=2(m+n)$ and is inclined 72° to the horizontal:

(5) $\qquad (-Y_b).2(m+n)\sin18^\circ-(-X_b)2(m+n)\cos18^\circ-2W(m+n)\sin18^\circ=0.$

Using (4) to solve successively (1), (5), and (2) we obtain

$$Y_b=-3W,\quad X_b=(W+Y_b)\tan 18^\circ=-2W\tan 18^\circ,$$
$$X_a=2W\tan 18^\circ-5W\cot 54^\circ.$$

Putting these values into (3) and dividing by $2W$ we get

$$(6)\qquad -2\tan 18^\circ.n.\cos 54^\circ-3W.n\sin 54^\circ+(m-n)\sin 54^\circ$$
$$+(5\cot 54^\circ-2\tan 18^\circ).m\cos 54^\circ=0.$$

Multiplying (6) by $\sin 54^\circ$, collecting m and n, and later employing the identities $2\sin^2 x=1-\cos 2x$, $2\cos^2 x=1+\cos 2x$, and $2\sin x\ \cos x=\sin 2x$, we obtain:

$$m[\sin^2 54^\circ-2\sin 54^\circ\cos 54^\circ\tan 18^\circ+5\cos^2 54^\circ]$$
$$=n[4\sin^2 54^\circ+2\sin 54^\circ\cos 54^\circ\tan 18^\circ],$$
$$m[3+2\cos 108^\circ-\sin 108^\circ\tan 18^\circ]=n[2-2\cos 108^\circ+\sin 108^\circ\tan 18^\circ],$$
$$\text{or, } m[3-3\sin 18^\circ]=n[2+3\sin 18^\circ],$$
$$\text{whence,}\quad m:n=2+3\sin(\pi/10):3-3\sin(\pi/10).$$

The proposed result follows immediately from this one, if we recall from geometry that the half-side of the regular inscribed decagon$=R(\sqrt{5}-1)/4$, so that $\sin 18^\circ$ satisfies the equation

$$(4\sin x+1)^2=5 \text{ or } 4\sin^2 x+2\sin x=1.$$

For from this we obtain $\sin x(3\sin x+2)=1-\sin^2 x$, or $2+3\sin x:3(1-\sin x)::1+\sin x:3\sin x$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

343. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

A, on contracting to execute a piece of work for \$300 and finding after working alone one day that he had finished but 1% of the entire work, engaged B to assist him at the beginning of the second day, with the understanding, that B on each day was to do 6% as much work as had been completed previously, while A each day was to do an amount of

work equal to 1% of the unfinished work at the close of the day before. At the completion of all the work the $300 were divided between A and B in proportion to the amount of the work each had performed.

Required—(1) The number of days to do the work; (2) on which day would the daily earnings of A and B be the same; and (3) the amount of money each was paid under the agreement.

344. **Proposed by V. M. SPUNAR, Cleveland, Ohio.**

Given $x^7-5x^2y^4=-1506\ldots(1)$, and $y^5-3xy=103\ldots(2)$; find the values of x and y.

GEOMETRY.

374. **Proposed by PROF. R. C. ARCHIBALD, Brown University.**

The locus of the middle points of chords, of a conic, which all pass through a fixed point P, is a conic. In general, four chords equal to a given length K can be drawn through P. Show that the middle points of these equal chords lie on a circle whose center is independent of K.

375. **Proposed by C. N. SCHMALL, New York City.**

From a point P on a circle there are drawn three chords PA, PB, PC. Show that the circles described on these chords as diameters intersect again in three collinear points.

376. **Proposed by S. LEFSEHETZ, East Pittsburg, Pa.**

Inscribe in a given circle a quadrilateral, having given the three diagonals.

CALCULUS.

300. **Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.**

Solve the differential equation obtaining the complete primitive:

$$(x^2+x^2y+2xy-y^2-y^3)dx+(y^2+xy^2+2xy-x-x^3)dy=0.$$

301. **Proposed by C. N. SCHMALL, New York City.**

Show that the volume of the surface,

$$\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}+\left(\frac{z}{c}\right)^{\frac{2}{5}}=1, \text{ is } \frac{100\,\pi\,abc}{3.7.11.13}.$$

BOOKS AND PERIODICALS.

College Algebra. By Schuyler C. Davidson, Sc. D., Professor of Mathematics in Indiana University. 8vo. Cloth sides and leather back, xiv+243 pp. Price, $1.50 net. New York: The Macmillan Co.

This book, we are told in the preface, is not written for the mathematician but for students wishing to know the elements of ordinary algebra. For this reason, the book is

not exhaustive, many subjects belonging to a treatise on Algebra being omitted. The first chapter deals with the natural number system and the second with rational numbers. The discussion carried on in these two chapters is admirable, but is, in our judgment, unintelligible to the average teacher into whose hands the book is intended to fall. However, much good may be done for this class of teachers by putting before them something lying outside the familiar path of their limited experience.

A sufficient number of problems are given to illustrate the underlying theory. The book will prove serviceable in classes whose previous preparation has been satisfactory. F.

Wentworth's Plane Geometry. Revised by David Eugene Smith. 8vo. Cloth, vi+287 pages. Price, 80 cents. Boston and Chicago: Ginn & Co.

This book is a revision of Wentworth's Plane Geometry, and has many points of superiority over the earlier editions. For example, in the introduction emphasis is laid on the instruments of geometry and practical exercises are given. The book closes with an article on geometric recreations and one on the history of geometry. F.

A Text-Book on Advanced Algebra and Trigonometry With Tables. By William Charles Brenke, Ph. D., Associate Professor of Mathematics in the University of Nebraska. 8vo. Cloth, vii+345 pages. Price, $2.00. New York: The Century Co.

The author of this book believes that the presentation of the two subjects, Algebra and Trigonometry, in a correlated manner, is more satisfactory than to take up the two subjects alternately. For that reason he has written this text. In it are to be found a fairly complete treatment of the ordinary subjects usually studied in the earlier part of a mathematical course. An introduction to the Differential Calculus is also inserted by making use of the derivative. To those teachers who share the author's belief, the book will be found serviceable. F.

Lectures on the Theory of Elliptic Functions. By Harris Hancock, Ph. D. (Berlin), Dr. Sc. (Paris), Professor of Mathematics in the University of Cincinnati. Vol. 1 Analysis. First Edition. First Thousand. Large 8vo. Cloth, xiii+498 pages, 76 figures. Price, $5.00. New York: John Wiley & Sons.

This is by far the most exhaustive and scholarly work that has thus far been published on this subject in America. It is the purpose of the author to present the Theory of Elliptic Functions in three volumes, which are to include the three following phases of the subject, viz: Vol. I, *Analysis*; Vol. II, *Applications to Problems in Geometry and Mechanics*; and Vol. III, *General Arithmetic and Higher Algebra.* In the development of the subject, the author places the Theory of Weierstrass side by side with the Theory of the older writers and many of the formulæ derived by him are contrasted with the corresponding formulæ of the earlier writers. The general theory is treated by means of Riemann surfaces, thus showing the intimate relations between the theory of Weierstrass and his predecessors.

It is to be hoped that the author will soon present the other two volumes to the American mathematicians, who are already greatly indebted to him for this first volume. F.

The Monist for July contains three important mathematical articles. The first is Mathematical Creation, by Henri Poincare; the second, The Construction of Magic Squares and Rectangles by the Method of "Complimentary Differences," by W. S. Andrews; the third, Magic Circles and Spheres, by Harry A. Sayles.

Practical Algebra, First Year Course. By Joseph V. Collins, Professor of Mathematics, State Normal School, Stevens Point, Wisconsin. 8vo. Cloth, 301 pages. Price, $1.00. New York and Chicago: The American Book Co.

This book is an abridgment of the author's Practical Algebra which appeared two years ago and which was noticed in the Monthly at that time. Human interest is added to this text by the insertion of portraits of Newton and Descartes. F.

Shop Problems in Mathematics. By William E. Breckenridge, Chairman of the Department of Mathematics, Samuel F. Mersereau, Chairman of the Department of Woodworking, and Charles F. Moore, Chairman of the Department of Metal Working, in Stuyvesant High School, New York City. Cloth, 12mo, 280 pages, illustrated. Price, $1.00. Boston, New York, Chicago: Ginn & Co.

"This book aims to give a thorough training in the mathematical operations that are useful in shop practice, e. g. in Carpentry, Pattern-Making, Foundry Work, Forging, and Machine Work, and, at the same time, to impart to the student much information in regard to shops and shop materials. The mathematical scope varies from addition of fractions to natural trigonometric functions. Problems are graded from simple work in board measure to the more difficult exercises of the machine shop. All problems are based on actual experience. The slide rule is treated at length. Short methods and checks are emphasized. The book should be useful in any schools where there are shops: i. e. in the upper grades of elementary schools for a review course in Practical Mathematics; in Manual Training High Schools as a supplementary book of problems all through the mathematical course and in the shops; in Trade Schools as a text-book either in the mathematics classroom or in the shop; in Normal Schools; in Apprentice Schools, and in the classes of the Y. M. C. A."

Theoretical Mechanics. By Percy F. Smith, Professor of Mathematics in the Sheffield Scientific School, Yale University, and William Raymond Longley, Assistant Professor of Mathematics in the Sheffield Scientific School, Yale University. 8vo. Cloth, 288 pages. List price, $2.50. New York and Chicago: Ginn & Co.

This book is intended for use in courses in mechanics which, as in many colleges and technical schools, are based upon the calculus. For the convenience of the student, formulas from analytic geometry and the calculus, and a table of integrals, are included. The first chapter deals with centers of gravity and moments of inertia. This is followed by chapters on kinematics and kinetics of a particle (including impact), motion in various fields of force (constant field, central field, harmonic field), kinetics of a system of particles, potential, motion in a resisting medium, dynamics of a rigid body, including uniplanar motion, and equilibrium of coplanar forces.

Attention is called to the following special features of the book: 1. The fundamental problem—to determine the motion due to a given force under given initial conditions—is thoroughly discussed. The equations of motion obtained by integration of the force equations have, however, been studied in a previous chapter, and the student is therefore cognizant immediately of the significance of his results. 2. Emphasis is laid everywhere in the solution of problems upon the general application of the force equations, the energy equation, and the impulse equation. 3. The problems are carefully selected, and numerous illustrative examples are worked out in the text.

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ATTEMPTS MADE DURING THE EIGHTEENTH AND NINETEENTH CENTURIES TO REFORM THE TEACHING OF GEOMETRY.*

By FLORIAN CAJORI, Colorado Springs, Colorado.

BIBLIOGRAPHY.

We have found the following six books bearing on the history of the teaching of geometry most useful in making this compilation:

1. V. BOBYNIN — "Elementare Geometrie," being Chapter XXII. in Cantor's *Vorlesungen über Geschichte der Mathematik*, Vol. IV, Leipzig, 1908, pp. 321-402. (Covers second half of eighteenth century.) Referred to as "Bobynin."

2. F. KLEIN — *Elementarmathematik vom Hoeheren Standpunkte aus, Theil II; Geometrie.* Leipzig, 1909, pp. 433-515. Referred to as "Klein."

3. J. PERRY — *Discussion on the Teaching of Mathematics.* British Association Meeting at Glasgow, 1901, London. Referred to as "Perry."

4. H. SCHOTTEN — *Inhalt und Methode des Planimetrischen Unterrichts.* Leipzig, 1890. Referred to as "Schotten."

5. M. SIMON — *Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert.* Leipzig, 1906. Referred to as "Simon."

6. A. W. STAMPER — *A History of the Teaching of Elementary Geometry,* New York, 1906. Referred to as "Stamper."

Other useful sources of information on the history of the teaching of geometry are as follows:

1. *Reports of the Association for the Improvement of Geometrical Teaching* (in England). The Association now calls itself "The Mathematical Association" and its present organ is the *Mathematical Gazette.*

2. *Zeitschrift für Mathematischen and Naturwissenschaftlichen Unter-*

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*This article is a part of the report of the National Committee of Fifteen on a Geometry Syllabus. The Committee has been at work for nearly two years under the joint auspices of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences. The Committee is not yet ready to present its report but feels that this historical setting prepared by Professor Cajori should be in the hands of mathematical teachers at once. THE EDITORS.

richt, Leipzig und Berlin. Formerly called "Hoffmann's Zeitschrift," now "Schotten's Zeitschrift."

3. *L'Enseignement Mathématique. Revue internationale.* Paris.
4. *Nature* (London). See Indexes for "Geometry."
5. G. LORIA — *Vergangene und Künftige Lehrplane.* Deutsch von H. WIELEITNER, Leipzig, 1906.
6. G. LORIA — *Della varia fortuna di Euclide,* Roma, 1893,
7. DODGSON — *Euclid and His Modern Rivals,* London, 1885 (2. ed.)
8. R. FRICKE. "Ueber Reorganisationsbestrebungen des Mathematischen Elementarunterrichts in England." *Jahresber. d. deutsch. Math. Vereinigung,* Vol. 13, 1904, p. 283, etc.
9. KLEIN-SCHIMMACK, *Vortrage über den mathematischen Unterricht an den Hoeheren Schulen,* Leipzig, 1907.
10. *Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons.* Paris, 1903.
11. A. W. Stamper's list of references at the end of his book.
12. Histories of mathematics.

FRANCE.

France began to maintain a critical attitude toward Euclid as a geometrical text-book for beginners as early as the time of Petrus Ramus (1580). Ramus treated geometry as the art of accurate measurement. In the eighteenth century this spirit of independence was intensified by the publication of Clairaut's *Elêmens de Gêometrie* (1741), in which surveying and other practical matters received marked attention. In the latter half of the eighteenth century Euclid ceased to be used as a text-book in France.

Williamson, in his edition of Euclid, 1781, criticises Clairaut as follows: "Elements of geometry carefully weeded of every proposition tending to demonstrate another; all lying so handy that you may pick and choose without ceremony. 'This is useful in fortification;' 'you cannot play at billiards without this.' 'You only look through a telescope like a Hottentot until this proposition is read,' with many such powerful strokes of rhetoric to the same purpose. And upon such terms, and with such inducements, who would not be a mathematician? Who would go to work with all that apparatus which I have described as necessary for understanding Euclid, when he has only to take a pleasant walk with Clairaut upon the flowery banks of some delightful river, and there see, with his own eyes, that he must learn to draw a perpendicular before he can tell how broad it is?" About 1836 De Morgan remarks that these arraignments are not "without their force, when directed against experimental geometry as an ultimate course of study, [but] lose their ironical character and become serious earnest, when applied to the same as a preparatory method." De Morgan strongly favors a geometry like Clairaut's as a preparatory course.

The critical attitude of Ramus and Clairaut toward the *Elements* of

Euclid brought to the mind of D'Alembert the questions: What are the elements of a science? What should be the contents of a book called elements? D'Alembert gives his answers in two articles, "Elémens des sciences," and "Des élémens de géometrie" in the *Encyclopédie méthodique* (about 1784).*

D'Alembert distinguishes between two kinds of elements of a science:

(1) If all truths or theorems of a science which are the foundation for all others, are brought together, so that these truths or theorems potentially comprise the whole science, then these constitute, when properly coordinated, the *elements* of the science. In geometry, such elements embrace not merely the principles of mensuration and the properties of plane figures, but also the application of algebra to geometry, and the differential and integral calculus in its application to curved lines.

(2) The elements of a science may be defined also as comprising those truths or theorems which treat the subject matter in the simplest way, and which constitute, together with their deductions, a detailed study of the simplest parts of the science. By the elements of geometry, elements of this kind are usually meant; they include only the properties of plane figures and the circle.

Dissatisfied with the elements of geometry known in his day, D'Alembert sets up the following demands which such texts should fulfill:

(1) The text should develop the subject along the path pursued by the discoverers of the science; so as to show the truths in their natural relations to each other.

(2) The usual division of the subject into longimetry, planimetry and stereometry does not provide for the circle and sphere, and is therefore inadequate. The division into plane geometry and solid geometry, D'Alembert does not consider at all. He suggests the division into the geometry of the straight lines (considered with respect to position and relative magnitude) and circles, the geometry of surfaces and the geometry of solids. The straight line and circle must be taken up together. The circle renders immense service in considering the position of lines. The measurement of angles by circular arcs and the principle of congruence constitute the basis of the first part of the geometry of lines, upon which other theorems of this part rest. The second part in the geometry of the straight line has as its fundamental theorem the one on the section into proportional parts of two sides of a triangle by a line parallel to the third side. This involves incommensurables.

(3) Incommensurable relations must be treated by the apagogic method, according to which it is shown that one ratio cannot be greater or smaller than a certain other ratio, hence it must be equal to that other ratio. He uses this for the following reasons: Incommensurable magnitudes in-

Encyclopédie methodique, Mathématiques I, 617-625; III, 133-136. We have used the Italian translation of this dictionary, Padova, 1800, and also a full abstract of these articles, given by Bobynin. See Bobynin, p. 325, etc.

volve the idea of the infinite and therefore, he claims, cannot be treated by any direct method. Notwithstanding this difficulty presented by incommensurable lines, he maintains that they should be taken up early in geometry, because of their importance. He states that the whole theory of incommensurables demands only one theorem, concerning the limits of quantities, viz: "Magnitudes which are the limits of one and the same magnitude, or magnitudes which have one and the same limit, are equal to each other." In the geometry of the circle, of surfaces and solids, he feels that the method of exhaustion or that of limits should be used.

(4) A suitable text on the elements of geometry can be prepared only by a mathematician of the first rank. D'Alembert complains that most elementary geometries are written by men of little ability.

(5) To lay down definitions at the beginning without any analysis of the subject is not only contrary to sound philosophy but contrary to the natural march of thought.* Axioms are useless.

Ideas similar to those of D'Alembert are embodied in a text on geometry by Louis Bertrand of Genève,† who in Berlin had been close to Euler. Bertrand's book antedated D'Alembert's articles in the Encyclopédie Méthodique. Like D'Alembert he divides geometry into three parts: (1) Geometry of line and circle, (2) Measurement of parts of a plane bounded by straight lines and circles, (3) Measurement of curved surfaces and solids. Bertrand ignored the classification of geometry into plane and solid. His second theorem is: "When two planes intersect, their common section is a right line." The straight line and circle are taken up together at the beginning as D'Alembert would have it. The incommensurable case is treated by the *reductio ad absurdum* method. In the latter part of the geometry he uses also the method of exhaustion. Bertrand reduces the number of theorems, in one instance, by replacing theorems on the mensuration of prisms, pyramids, cylinders, cones, and spheres by the corresponding problems.

Bertrand's work was published in two unwieldy volumes and had little sale, yet exercised some influence, particularly upon Lacroix, whose *Cours de Mathématiques*, published at the close of the eighteenth century, has been used until recently. Lacroix divides his geometry into geometry of the plane and geometry of space, and does not follow D'Alembert closely. According to Lacroix there are only two kinds of theorems that should find a place in an elementary geometry: (1) Theorems necessary for the comprehension of the line of argument, developed synthetically. (2) Theorems which grow out of the practical operations in geometry (drawing and measuring). He objects to placing all axioms at the beginning, believes in the omission of the definition of an angle, favors "a straight line is the

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*See "Axiome" and "Courbe" in *Encycl. Méth.*

†*Dévelopment nouveau de la partie élémentaire des mathématiques*, Genève, 1778.

shortest path between two points'' as growing out of the child's experience, and uses the apagogic method for incommensurables.

Another author of note was Bézout, who followed D'Alembert's plan quite closely, but was criticised for his lack of rigor and for his endeavor to lighten the work of the examiner as well as of those being examined.*

The most celebrated work on elementary geometry is that of Legendre (1794). He came nearest to fulfilling D'Alembert's requirement that the elements be written by a mathematician of the first rank. He does not follow D'Alembert's plan for a book on geometry, nor does he heed the philosophic demand that the author should follow the path of the originators of the science. Impressed by the lack of rigor in the works of his day, he aims at greater rigor and approaches closer to Euclid than his predecessors had been. He does not divide geometry in the manner of D'Alembert and Bertrand. Like Euclid, Legendre begins with definitions and axioms. The first four chapters are given to plane geometry, the last four to solid. The first book treats of the equality of angles and triangles, the second of the circle and the measurement of angles, the third of proportional figures, the fourth of regular polygons and the measurement of the circle. Legendre uses in measurement the terms *equal* and *equivalent.* He uses the *reductio ad absurdum* method for incommensurables and the method of exhaustion for curved lines.

What was it that made this book so successful? In the first place must be mentioned his great clearness of exposition and his attractive style. A great advance of Legendre over Euclid was the fuller treatment of solid geometry. He leans less toward logic and more toward intuition than does Euclid. In place of Euclid's famous fifth book on incommensurables, Legendre borrows rational and irrational numbers from arithmetic, even though in arithmetics no scientific treatment of those subjects was given in his day. A theorem true for rationals is assumed to be true for irrationals. Thus, if $A:B=C:D$, then $AD=BC$ in all cases. Klein says that this is in accordance with the practice of the best mathematicians of his day, that even Lagrange works out the expansion of $(x+h)^n$, when n is rational and assumes the results thus obtained to be true for irrational values of n. Legendre stands for a fusion of geometry, not only with arithmetic, but also with trigonometry. As late as 1845 Legendre's geometry still contained trigonometry, but as Klein remarks,† the trigonometry and the practical application of geometry were gradually filtered out. Comparing A. Blanchet's edition of 1876 with an edition of 1817, we find also that the twelve ''notes'' on topics of elementary geometry, covering 55 pages in the older edition, are omitted in the later edition. The later edition has a somewhat fuller treatment of solid geometry and a list of exercises in original proofs, loci and constructions. Other notable changes were made in the

*Bobynin, p. 355.
†Klein, p. 470.

1845 edition by J. B. Balleroy and A. L. Marchand. They state that Legendre uses the *reductio ad absurdum* method to excess, a method which "convinces but does not satisfy the mind." Legendre's text is, however, left intact, alternative proofs being given in notes at the end. These alternative proofs, as well as the proofs given in the modified text of the 1876 edition, are rough applications of the theory of limits.

During the first half of the nineteenth century, and even later, the works of Legendre, Lacroix and Bézout were used extensively in France. In later editions less stress was laid upon practical applications and numerical computation. Otherwise few changes occurred. In general, school organization, based on the regulations of the time of Napoleon I., was quite fixed in France until 1870. France has a rigid centralization of authority in education. If the "Conseil d' instruction supérieure" decides upon a change, the whole country adopts it at once. As compared with the German, the French teacher has little individual freedom. France is a country with a "system of revolutions from above."* Since 1870 the movement has been toward greater individual freedom. The later tendencies in geometry are imaged in the work of Rouché and de Comberousse, which contains a large amount of new material and meets the demands of the two year courses of the *classes de mathêmatiques spêciales* during which as much as sixteen hours per week are given to mathematics and a degree of specialization is allowed in preparation for university courses, as in no other country. In 1902 and 1905 official courses of study were adopted in France in which greater stress is laid upon graphic representation, the idea of a variable and a function, and upon the practical applications of mathematics. This new tendency is mirrored in the geometry of E. Borel, a remarkable book, in which the practical receives due emphasis and in which intuition meets with fuller recognition. With Borel the concept of motion is prominently used. There is an introduction of eight pages on the use of the ruler, compasses, and protractor, and ten pages on the mensuration of surfaces and solids, treated empirically. Applications are skillfully interwoven with theory, throughout the book. He has well selected practical exercises involving symmetry, the nets of regular polygons, the use of pulleys, and so on. Algebraic geometry and the development of metric properties come last in the book. He introduces the rudiments of trigonometry. The usual division into plane geometry and solid geometry is not rigidly maintained.

A parallel and somewhat different tendency in France is seen in the geometry of Ch. Méray of Dijon, which was first brought out in 1874 but has only in recent years received much attention. Méray represents the severely logical mode of exposition;† he uses in his proofs no fact of observation which has not been previously set down in an axiom; he formulates a complete list of axioms, but introduces each only when it is needed;

*Klein, p. 457.

†Klein, p. 475.

nor does he aim to limit their number to a minimum. Characteristic of Méray is the complete fusion of plane and solid geometry, and the use of motion. Prepared under the influence of Méray is the recent (1908) geometry of C. Bourlet.

Influenced by the Perry movement in England and America, France is experimenting on the laboratory method of instruction.* A laboratory was founded by J. Tannery and E. Borel. Recently there has been considerable discussion in France on the question whether in laying the foundations to geometry, *motion* should be used or not. The defenders of a static theory of parallels claim that motion cannot be visualized on the board, rendering intuition more difficult. The defenders of the kinematic theory advocate the use of movable figures.†

GERMANY.

Klein‡ expresses surprise that, during the Renaissance, Euclid should have come to be looked upon as a text suitable for the first instruction in geometry. Perhaps the reason for this attitude toward Euclid lies in the fact that geometry was first taken up in the universities by students of maturer years. As geometry came gradually to be taught to younger and younger pupils, Euclid was still retained. Thus the misconception arose that Euclid was a suitable geometrical text for young boys.

While D'Alembert formulated his ideas on elementary geometry in France, A. G. Kastner evolved in Germany a type of his own, in his work, *Anfangsgründe der Arithmetik, Geometrie, Trigonometrie und Perspectiv*, Goettingen, 1758. Kastner begins with definitions and axioms in Euclidean style, develops the geometry of the plane (69 pages) and ends this part with practical applications (47 pages). The second part of the geometry begins with the geometry of space (60 pages), continues with 31 pages given to plane trigonometry and its applications to the solution of triangles, and with 9 pages of practical geometry. Then follow spherical trigonometry and 24 pages on perspective. The method of exhaustion is used. It was the opinion of Kastner that "the newer works on geometry lose the more in clearness and thoroughness, the farther they depart from Euclid." He complains that modern authors, particularly the French, have departed from the ancient rigor, "to make the study of mathematics easier for people whose main occupation is not study, namely for soldiers."

Not without interest is W. J. G. Karsten's *Lehrbegriff der gesamten Mathematik*, in eight volumes, 1767-77, the first two volumes of which are given to arithmetic and geometry. Karsten begins with arithmetic, then proceeds to plane geometry, closing with simple arithmetical applications. He proceeds thereupon to solid geometry, returns to arithmetic, and gives

*Scholten, *Zeitschrift*, Vol. 40, 1909, pp. 444-5; *L'Enseignement Mathématique*, 11, p. 206.
†Schotten, *Zeitschrift*, Vol. 40, 1909, p. 445.
‡Klein, pp. 434, 435.

the rudiments of algebra with logarithms, followed by trigonometry and its applications to plane geometry. Finally are given the rudiments of spherical trigonometry and a fuller treatment of solids. Nowhere are heavy demands made upon the pupil. That this exposition was intended for students of university grade rather than those in the preparatory school, testifies to the low state of mathematical instruction in German universities of the eighteenth century. Close relation between arithmetic, geometry and trigonometry is also maintained in the works of J. G. Büsch (1776) and G. S. Klügel (1798), the aim being to make the subject easy of comprehension.

In the nineteenth century, until near its end, advanced mathematicians in Germany took little or no part in the improvement of the teaching of elementary mathematics. In geometry, Euclid's text was not usually taught, but the dogmatic method of Euclid was in vogue during the first half. About the middle of the century Euclid's order of the theorems came to be criticized as chaotic. It is interesting to see the Germans attack Euclid's order as arbitrary and the English defend it as the only order worthy of serious consideration. The grouping of theorems according to subjects came to be discussed in Germany.* The advocacy of object teaching by Pestalozzi, the championing of Pestalozzianism by Herbart, the attacks upon mathematical reasoning and particularly upon Euclid that were made by Schopenhauer† conspired to influence the teaching of geometry.

About 1860 the genetic method (called "heuristic" when the inventional side was emphasized) came to be discussed, which makes a plea of being a natural method, since it incites self-activity in the pupil. With the genesis of a theorem the pupil sees intuitively its inner relation to other theorems; he not only sees whence he came but also whither he is going; the reader of Euclid is blindfolded, so to speak, and then somehow transported to the next station. It is difficult to prepare text-books for the genetic method. The teacher by careful questioning one moment leads the student, the next moment follows him, and no one can foresee the exact path which this mode of advance will mark out. It is not strange, therefore, if many teachers proceeded heuristically while the texts retained mostly the dogmatic form.‡ Moreover, experience made it plain to teachers that the dogmatic statement of theorems has a high mnemotechnic value.§ While the genetic method in its pure form has not succeeded in establishing itself, it has exerted a strong influence by shifting the emphasis from the memorizing of proofs to the cultivation of originality and logical reasoning.

Another movement that sprang from the teachings of Pestalozzi and Herbart was the adoption of preliminary courses on observational geometry and drawing, about 1870. Such courses had been recommended long before this time. This movement was stronger in Germany than in England and

*Schotten, p. 11.
†Klein, p. 503.
‡Schotten, p. 96.
§Schotten, p. 13.

France. In their propædeutic courses the geometry of solids was to receive consideration and a taste of the genetic method was recommended. The pupils acquired dexterity in the use of ruler and compasses. Propædeutic courses have maintained their place to the present time.

Herbart made strong endeavors to remove the superstition that had arisen in early days when Euclid was placed in the hands of young and immature students, to the effect that mathematics could be learned only by a few pupils endowed with special gifts. According to his view the fault lies as a rule in the abstract character of the early instruction; the introduction of propædeutic courses and the greater emphasis upon "Anschauung" at all stages had shown that most students can master mathematics. Whether "amathematicians" do exist in rare instances, is a question which Klein refers to experimental psychologists for reply.*

A third movement agitated in Germany, was in favor of the introduction into elementary instruction of the concepts of the modern projective geometry. It originated about 1870.† The criticism was made that Steiner, Moebius and von Staudt had been so busy with their researches as to make no attempt to reform elementary instruction, and that text-book writers had ignored the researches of these great men. The leaders in this attempt to incorporate modern methods were Schlegel and Fiedler. A concomitant of this programme was the breaking down of the division of geometry into plane and solid, and the effort by the use of models, etc., to make geometry more concrete. To effect this reform, a number of texts by Schlegel, Müller, Kruse, Becker, Worpitzky, Henrici und Treutlein sprang into existence.‡ Aside from the production of interesting text books this agitation has had little success. The books in question were seldom used.§ Can it be that D'Alembert's dogma is, after all, based upon truth—the dogma that the historical order of development of geometry is the pedagogical order; that is, the easiest approach to the science for the young mind? Are the concepts of projective geometry more difficult to grasp than those of the older geometry, or did the texts just named overtax the pupils, and perhaps in other ways violate the demands of sound pedagogy?

Most interesting are the statistics gathered in Prussia in 1880 which showed the following distribution of geometrical texts: Kambly was used in 217 institutions; Koppe in 54; Mehler in 44; Reidt in 29, while 55 texts were used in one institution each. Kambly's "clever but unscientific book" was first issued in Breslau in 1850 and a few years ago reached the 101st edition in the revision by Roeder. Koppe was looked upon as an inferior work, yet it enjoyed great popularity. On the other hand, books like those of H. Müller and even Henrici und Treutlein seldom passed beyond the second edition. This most astonishing success of works considered as scien-

*Klein, p. 499.
†Schotten, p. 18.
‡Schotten, p. 19.
§Schotten, p. 20.

tifically inferior, requires explanation. Schlegel says* "that the quality of the books most widely adopted allows one to draw an inference respecting the scientific level of the instruction generally reached in that subject." But this remark considers merely one phase of this question. May not the mass of teachers have had a feeling or insight concerning text books which involved questions of intuition or other psychologic matters that the writers of the more scientific books overlooked? Simon† points out that until recently the German teacher, unlike the French, enjoyed complete freedom in teaching, and that small texts, like Kambly, allow his individuality much wider play.

Kambly's "Elementar-Mathematik" was made up of four parts: first, arithmetic and algebra; second, planimetry; third, plane and spherical trigonometry; fourth, stereometry. Of interest here, is the interpolation of trigonometry. We have before us Kambly's *Planimetrie*, 43d edition, Breslau, 1876. Among the points of popularity we mention the following:

1. The book contains only as much matter as a class can conveniently finish in one year. Skipping parts of a book, says Kambly, has a bad effect upon both pupils and parents.

2. The diction is clear and simple. Mathematical symbols are used freely. The setting of the type is such as to enable the eye more quickly to see the relations set forth.

3. The arrangement of the book is such as to allow the teacher much freedom. He may, for instance, omit incommensurables altogether, or else substitute for certain proofs in the regular text others given in the foot notes where rough proofs are found for incommensurable cases.

4. Easy arithmetical applications, original theorems, and original constructions are given at the end of the book, so that some, or all, may be conveniently taken or omitted, according to the preference of the teacher.

Koppe's *Planimetrie* made somewhat greater demands upon the powers of the pupil than did Kambly, but incommensurables were treated only in foot notes or in remarks following the proofs of theorems. In Lübsen's *Elementar Geometrie*, I have not been able to find a reference to incommensurables. It differs from Kambly and Koppe in having better figures and in having them on the page where they are needed, instead of the end of the book on separate sheets that unfold out. A clever feature in Lübsen are the practical applications introduced from the very beginning. How to run a straight line over undulating country by the use of poles, is explained in several diagrams on the first pages. Other figures show how to determine the distance between points on opposite banks of a river.

Since about 1890 the activity of Felix Klein of Goettingen, in mathematical reform, has been very great. For the first time since the death of Kastner, is the influence of university professors upon the teaching of ele-

*Schotten, p. 21.
†Simon, p. 25.

mentary mathematics in Germany beginning to be strongly felt. Among the defects of geometrical instruction, he points out the insufficient fusion of the various branches of elementary mathematics.* Thus, too little attention is given to drawing of solids and to projection, to the idea of motion in a figure to replace Euclidean rigidity, to the fusion of arithmetic and geometry, to the introduction of the coordinate representation of analytics. On the other hand, the construction of triangles from given data, is over emphasized,† as is also the study of the curious points and lines in the geometry of the triangle. This last criticism applies even more strongly to English text books.

Klein points out that modern demands in geometric teaching, *first,* emphasize the psychologic point of view,‡ which considers not only the subject matter, but also the pupil, and insists upon a very concrete presentation in the first stages of instruction, followed by a gradual introduction of the logical element; *second,* call for a better selection of the material from the view point of instruction as a whole; *third,* insist on a closer alignment with practical applications; *fourth,* encourage the fusion of plane and solid geometry, and of arithmetic and geometry.§

A piece of research of vital importance in the advanced study of geometry is the *Foundations of Geometry,* brought out in 1899 by Professor Hilbert of Goettingen.|| Though widely read by mathematicians, it has exerted no direct influence upon elementary teaching in Germany. It has been felt that this mode of treatment is not suitable for pupils first entering upon demonstrative geometry.

ITALY

Since the unification of Italy, great mathematical activity has existed in that country. Before that event, very different practices in geometrical teaching existed in different parts of the country.¶ In 1868 Cremona and Battaglini were members of a government commission to inquire into the state of geometrical teaching. They found it unsatisfactory, and the number of bad text books so great, and so much on the increase, that they recommended for classical schools the adoption of Euclid, an edition of which was brought out by Betti and Brioschi. Later other works of scientific merit replaced Euclid. Cremona's great emphasis upon projective geometry reached from the universities down into secondary schools. A typical work is that of A. Sannia and E. d'Ovidio, 1869, which uses the theory of limits and retains the division of geometry into plane and solid.

*Klein, p. 439.

†Klein, p. 442.

‡Klein, p. 435.

§Klein, p. 437.

||D. Hilbert, "Grundlagen der Geometrie" in *Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals in Goettingen,* Leipzig, 1890.

¶Simon, p. 43.

It stands closer to Euclid than to Legendre. The blending of plane and solid geometry, which received great emphasis in Italy, is typified in the *Elementi di Geometria* of R. de Paolis, 1884.

A very remarkable school came into being in Italy, the purpose of which is to render geometry still more rigorous than in the Euclidean text. Starting with a single basic concept, the point, all other concepts are to be logically developed. This movement is typified in the works of G. Veronese.* Of elementary works he has prepared *Nozioni Elementari di Geometria Intuitiva*, 1902, and *Elementi di Geometria*, 1904, the first of these being a propædeutic work. Demonstrative geometry is taken up in Italy with older pupils than in Germany and the United States; hence works of greater rigor can be used. Veronese endeavors to state all the necessary postulates of geometry, no matter how obvious, as for instance, "There exist *different* points," to make it plain that we do not consider a geometry in which only one point exists.† As regards the selection of material, Veronese confines himself mainly to that of Euclid, thus receding from the tendency of the School of Cremona. He avoids all fusion with arithmetic. Somewhat similar in character is the *Elementi di Geometria* of F. Enriques and U. Amaldi, 1905.

The effort at rigor, due to Veronese, has been intensified in the great school of Peano, which endeavors to eliminate all intuition. It seems that this school has influenced even elementary instruction and the teaching in technical schools.‡ This recent Italian emphasis upon extreme rigor has led to deplorable results with the less gifted pupils, and a reaction appears to be setting in. Under the leadership of Loria and Vailati a movement is on foot favoring greater emphasis upon intuition, the introduction of some modern geometrical notions, the fusion of geometry with arithmetic, and the concession to the demands for practical applications made by this age of industrial development. In fact, Italy is entering upon a reform much like that of Germany and France.§

ENGLAND.

Roger Bacon says that toward the close of the thirteenth century the definitions and a few of the theorems in geometry were studied by some pupils at Oxford.|| About 1570 Sir Henry Savile began to lecture at Oxford on Greek geometry, and in 1619 Briggs at Cambridge on Euclid. In 1665, Isaac Barrow at Cambridge prepared a complete edition of Euclid, which was the standard for fifty years. Gow says that "The seventy years or so, from 1660 to 1730, when Wallis and Halley were professors at Oxford, Bar-

*Klein, p. 482.

†Klein, p. 483.

‡Klein, p. 486.

§For additional details see W. Lietzmann's article in Schotten's *Zeitschrift*, Vol. 39, pp. 177-191; Vol. 40, pp. 227-228.

||Ball, *Mathematics at Cambridge*, 1889, p. 3.

row and Newton at Cambridge, were the period during which the study of Greek geometry was at its height in England."* In 1703, William Whiston became the successor of Newton at Cambridge. He brought out an edition of Tacquet's Euclid. Robert Simson's edition of Euclid first appeared in 1756. Simson was professor of mathematics at the University of Glasgow. In the universities of Great Britain, Euclid met with no competition. Ward's *Young Mathematician's Guide*, 1707, may have been used to some extent, but probably more for its arithmetic and algebra than for its geometry. Practical men, holding positions as excise officers, had to be familiar with practical geometry. For them practical treatises existed, some of which gave explanations of the slide rule. A departure from Euclidean rigor might be expected in the education of men for the army or navy. We have seen that Kastner criticised the French for making mathematics easy for men interested in war. England has had since 1722 an academy at Portsmouth where men spent one or two years studying navigation, drawing, etc. England has had also, since 1741, a military academy at Woolwich, where sons of noblemen and military officers were taught fortification, gunnery and mathematics. Among the mathematical professors at Woolwich, during the eighteenth century, were Thomas Simpson, John Bonnycastle and Charles Hutton, all three authors of text books including geometries. Hutton's works went through several editions in the first half of the nineteenth century. From this it is evident that Euclid did not hold universal sway in England. Yet the forces opposing him were utterly unable to dislodge him.

In 1822, Sir David Brewster brought out an English translation of Legendre's geometry. Did teachers rally in favor of the introduction of this text? We shall see that DeMorgan suggested the use of some parts of it on solid geometry; DeMorgan deplored that solid geometry was seldom or never taught before trigonometry. But otherwise we are not able to find any serious reference to this translation of Legendre.

During the second half of the eighteenth century England had come to be the only country where Euclid was practically the only geometrical text used. During the eighteenth century the average age of freshmen in the English universities was gradually increasing, and perhaps at this time, Euclid passed from the universities to the lower schools. There is no explicit proof, however, that in the great "Public Schools" Euclid was studied before the nineteenth century.†

Very recently‡ some interesting information has been published about one of the "public schools"—Christ Hospital—which paid more than usual attention to mathematics in the courses for boys preparing to enter the royal navy. It seems that as early as 1680 such boys were required to study the earliest parts of the first book of Euclid, the 10th, 11th and 12th

*Gow, *History of Greek Geometry*, Cambridge, 1884, p. 208.

†Stamper, p. 88.

‡"A School Course in Mathematics in the XVII Century" by W. W. R. Ball in the *Mathematical Gazette*, Vol. V, 1910, Part I, pp. 202-205.

propositions of the sixth book, and to learn arithmetic. Perhaps this represented all the theoretical mathematics taught, for Sir Isaac Newton, whose advice about changes in the course was sought, notes the following omissions: There was no "symbolic arithmetic," no "taking of heights and distances and measuring of planes and solids," no "spherical trigonometry," nothing of "Mercator's chart." In other "public schools" probably no courses in geometry were given during the eighteenth century. Says Stamper: "It was not until about the middle of the nineteenth century that the study of Euclid became common in the secondary schools of England."

It would be instructive to secure more information explainiug how it was possible for Euclid to maintain its supremacy as a text, when geometry was being transferred from the universities to the schools. What were the experiences of teachers in secondary schools with the Euclidean text? The desirability of modifying Euclid must have arisen early, for in 1795 John Playfair brought out a revised Euclid containing the first six books and adding the computation of π and a book on solid geometry drawn from modern sources. Playfair endeavored to give the geometry a form which would render it more useful. Euclid's fifth book, which had never been used successfully with beginners in geometry, as far as we can ascertain, was modified by Playfair by replacing Euclid's prolix explanations by the more concise language of algebra. But Playfair did not try to change the nature of the reasoning. Had there been a strong movement against Euclid in England at this time, Playfair would probably have joined it. In his review of Leslie's *Geometry* in the Edinburgh Review, Vol. 20, 1812, p. 79, he says: "A question has been sometimes agitated whether it is most advantageous, for the study of geometry, to possess a number of elementary treatises, or to have one standard work, like that of Euclid . . . the same lessons are not suited to every intellect, and on these accounts it may be of advantage that different elementary texts should exist. We are very much inclined to the latter opinion."

William George Spencer's unique booklet on *Inventional Geometry* was brought out about 1830 or 35, but "received but little notice" at that time. A noteworthy device for aiding the young mind through sensuous stimulus was the use of colored diagrams, suggested by Oliver Byrne, in his edition of Euclid, London, 1847. The failure of this book is doubtless due to the want of moderation in the use of colors.

The ablest writer on the teaching of elementary geometry during the first half of the nineteenth century in England, was Augustus DeMorgan. His articles published in the *Quarterly Journal of Education*, in 1831, 1832, 1833, display a pedagogical insight which would have prevented many calamities in English teaching, had his views been more promptly and widely accepted. Elsewhere we quoted DeMorgan's remarks on Williamson's criticism of Clairaut's geometry, which showed that DeMorgan firmly believed

in a preliminary course in Geometry, as an introduction to a logical course like that of Euclid. It will appear that England was the last country actually to introduce propædeutic courses in elementary instruction.

DeMorgan did not hesitate to recommend radical changes in Euclid. Here is what he said in 1831, in an article in the *Quarterly Journal of Education*, entitled "On Mathematical Instruction:"

"With regard to the fifth book of the Elements, we recommend the teacher to substitute for it the common arithmetical notions of proportion. Admitting that this is not so exact as the method of Euclid, still, a less rigorous but intelligible process is better than a perfect method which cannot be understood by the great majority of learners. The sixth book would thus become perfectly intelligible."

Two years later, in an article in the same journal "On the Methods of Teaching the Elements of Geometry," DeMorgan dares to suggest that certain parts of Legendre might be profitably substituted for parts of Euclid. "The eleventh book of Euclid may, in our opinion, be abandoned with advantage in favour of more modern works on solid geometry, particularly that of Legendre, which the English reader will find in Sir David Brewster's Translation." In the same article DeMorgan gives utterance to a difficulty experienced by young students, which has been referred to by many writers in different countries, the *reductio ad absurdum.* DeMorgan says: "The most serious embarrassment in the purely reasoning part is the *reductio ad absurdum*, or indirect demonstration. This form of argument is generally the last to be clearly understood, though it occurs almost on the threshold of the elements. We may find the key to the difficulty in the confined ideas which prevail on the modes of speech there employed." As regards the difficult fifth book, DeMorgan said, in 1833, "We would say to all, teach the fifth book, *if you can*; but we would have all remember that there is an *if*." In another place he adds: "We strongly suspect that Euclid, as studied, does as much harm as good." To the credit of teachers be it said, that the fifth book was quite generally omitted. But DeMorgan's activity in this line did not end here. In 1836 he published *The Connexion of Number and Magnitude; An attempt to explain the fifth book of Euclid.* For fifty years this tract was not duly appreciated; later it began to wield a wide influence; it is on this tract that the substitute for the fifth book given in the Syllabus of the Association for the Improvement of Geometrical Teaching is modeled; it is on this tract that the revised fifth book in the more recent editions of Euclid by Nixon and by Hall and Stevens is based.

The need of modifying the text of Euclid is brought out by DeMorgan in the *Companion to the British Almanac* of 1849, page 20, as follows: "If the study of Euclid have been almost abandoned on the Continent, and have declined in England, it is because his more ardent admirers have insisted on regarding the accidents of his position as laws of the science."

How little influence DeMorgan's views wielded in England before

about 1870 as regards the revision of Euclid's fifth book and the study of solid geometry, appears from the fact that the most popular edition of Euclid for many years, was the one brought out in 1862 by Todhunter. This author reproduces Simson's text, though he greatly assists the pupil in overcoming the difficulties by breaking up the demonstrations into their constituent parts. In an Appendix are given notes, supplementary propositions and original exercises. Todhunter was quite out of sympathy with the purposes of the Association for the Improvement of Geometrical Teaching.*

Opponents of Euclid existed in England at all times. Thus in 1860 W. D. Cooley brought out a rival text. Eleven years later he expressed himself regarding this venture as follows:†

"In 1860 there was published for me, by Messrs. Williams and Norgate, a little volume entitled, *The Elements of Geometry Simplified and Explained,* adapted to the system of empirical proof, and of exhibiting the truth of theorems by means of figures cut in paper. It contains in 35 theorems the quintessence of Euclid's first six books, together with a supplement not in Euclid. There was no gap in the sequence or chain of reasoning, yet the 32d and 47th propositions of Euclid were, respectively, the 3d and 17th of my series. This book proved a failure, for which several reasons might be given, but it will be sufficient here to state but one, namely, that it came forth ten years before its time."

The reformers found a champion in Sylvester, who in 1869, before Section A of the British Association exclaimed: "I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister (natural science) could not fail to impart, short roads preferred to long ones, Euclid honourably shelved, or buried 'deeper than e'er plummet sounded' out of the school boy's reach. . . ." The reform forces finally organized themselves, in 1871, into the "Association for the Improvement of Geometrical Teaching" (A. I. G. T.).

It is a curious circumstance that England's great mathematician, Arthur Cayley, opposed this reform movement. His admiration for Euclid was so ardent that he even expressed a preference for the original treatise without Simson's additions. In the opinion of Langley, Cayley "overshot the mark and his opposition told in favor of the Association."‡

The second *Report* of the A. I. G. T. recommended practical exercises in geometrical construction, easy originals and numerical examples. Two years were given to the preparation of the Geometrical Syllabus on proportion. A double syllabus was prepared: A Syllabus on Geometric Constructions, and a Syllabus on Plane Geometry. Most of DeMorgan's suggestions§ on the revision of the fifth book of Euclid were adopted.

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*See *Conflict of Studies,* by Todhunter, London, 1873.

†Nature, Vol. 4, 1871, p. 486.

‡*Fifth Report of the A. I. G. T.,* p. 21.

§*Companion to the British Almanac,* 1849, pp. 5-20.

This Society, after long labors, finally issued a substitute text, *The Elements of Plane Geometry.* This was not used at home, but was used with success in the British colonies. Klein expresses himself, as follows, in regard to it: "This is essentially merely a smoothed down and polished presentation of the first six books of Euclid's elements; thus the rough places at the beginning of the first book . . are removed by a consistent use of the concept of motion, but in general the sequence and the contents of Euclid are adhered to, in deference to the examinations. It is therefore only a tame reform, that is here attempted; nevertheless, it has met with sharp opposition by the adherents of the old English system. As proof of this, I refer to an amusingly written book of Dodgson, *Euclid and His Modern Rivals.*" Here Euclid comes out victorious, and all reformers, particularly Legendre and members of the A. I. G. T., are put to the rout. J. M. Wilson's *Elementary Geometry*, 1st edition, 1869, came in for a large share of the criticism. At Oxford, where Dodgson had given instruction in geometry for many years, this same Wilson had, at one time, read a critical paper before the Mathematical Society, on "Euclid as a Text Book of Elementary Geometry." Wilson was a prime mover in the organization of the A. I. G. T.

Since about 1870, many editions of Euclid have been printed containing revisions with the object of better adapting Euclid to school use. They exhibit all possible gradations of departure from the original text. There appeared sequels to Euclid like that of F. Casey. Professor Klein expresses himself in regard to these as follows: "The necessity has been felt to consider modern research, going beyond Euclid; this has been done by pressing it by force into the rigid Euclidean form, whereby a good part of the modern spirit is, of course, lost."*

During thirty years the A. I. G. T. appeared to have accomplished comparatively little. It had secured the concession that proofs different from Euclid's shall be accepted in examinations and had brought about a sentiment favoring some modification and enrichment of the Euclidean text. In reality, it had accomplished much more, for it had prepared the way for the great agitation of 1901, known as the Perry Movement, which called for a complete divorce from Euclid. The discussion of the teaching of mathematics at the Glasgow meeting of the British Association marks an epoch. The following are suggestions and criticisms that were contained in Perry's Syllabus.†

1. Experimental geometry and practical mensuration to precede demonstrative geometry. Use of squared paper. Rough guessing at lengths and weights to be encouraged.
2. Some deductive reasoning to accompany experimental geometry.
3. More emphasis on solid geometry; this subject has been postponed too long.

*Klein, p. 447.

†*Discussion on the Teaching of Mathematics*, edited by John Perry, 1901, p. 97.

4. Adoption of coordinate representation in space.

5. The introduction of trigonometric functions in the study of geometry.

6. Emphasis upon the utilitarian parts of the subject.

7. Examinations conducted by any other examiner than the pupil's teacher are imperfect examinations.

In criticism of previous practices, Perry held that a boy should be educated through the experiences he already possesses, and should be allowed to assume the truth of many propositions. He held that the teacher must recognize that boys take unkindly to abstract reasoning. He criticised Oxford because, for the pass degree there, two books of Euclid must be memorized, even including the lettering of figures, no original exercises being required. In the discussion that followed, all favored the preliminary experimental course and some advocated a second experimental course to accompany Euclid. Hudson and Forsyth still believed in maintaining the Euclidean sequence of theorems. Minchin declared Euclid's order bad. S. P. Thompson and MacMahon favored the retention of Euclid. Miall did not see why we should have a recognized geometry any more than one arithmetic, or one trigonometry. Minchin, Magnus, Pressland, Workman and Lamb declared themselves against Euclid as a text book.

The immediate result of Perry's address of 1900, at Glasgow, was the appointment of two committees, one of the British Association and the other of the Mathematical Association. The former committee confined its work to the more general aspects of geometrical teaching. The latter, which was composed mainly of school masters, formulated a set of detailed recommendations, which were published in the *Mathematical Gazette* of May, 1902. They include an experimental introductory course, requiring the use of instruments, practical measurement and numerical work. In the formal study of geometry is recommended the retention of Euclid as a framework, the admission of hypothetical constructions, definitions not to be taught *en bloc*, the omission of incommensurables in the ordinary school course, the use of algebra in the treatment of areas.

The Perry laboratory method has led to the preparation of some severely practical works, but as Lodge says, Perry "over emphasized fact divorced from principles." A middle ground has met with greater favor. The plans recommended by the Mathematical Association have been embodied very successfully by Godfrey and Siddons in a text book entitled, *Elementary Geometry, Practical and Theoretical* (Cambridge University Press, 1904). The recommendations of the Mathematical Association have met with favor among teachers, and the general effect has been beneficial. A circular issued in 1908-1909, by the Board of Education, on *The Teaching of Geometry and Graphic Algebra*, showed the wide departure made since the beginning of the twentieth century. We quote two sentences: "Axioms and postulates should not be learnt or even mentioned." "It

should be frankly recognized that unless the power of doing riders has been developed, the study of the subject is a failure."*

The greatest obstacle to reform in England has been the system of examinations. After thirty years of failures the Mathematical Association, at last, has been remarkably successful in persuading examining bodies to give up their insistence upon Euclid, and now Euclid's proofs and arrangement are no longer required by the universities. "Any proof of a proposition will be accepted which appears to the examiners to form a part of a logical order of treatment."

The United States of America.

During the seventeenth century, arithmetic and geometry received some attention in the last year of the college course at Harvard College. In 1726 Alsted's *Geometry* is mentioned as a text book studied by Harvard seniors, but as soon as geometry came to receive serious attention in American colleges, Euclid became the text used. The first mention of Euclid that we have seen, at Yale, is in 1733; at Harvard, in 1737. In the latter part of the eighteenth century, geometry was taught to lower classmen. According to a member of the Harvard class of 1798, "the sophomore year gave us Euclid to measure our strength." In 1801 Professor Webber said, "A tutor teaches in Harvard College Playfair's Elements of Geometry."

In 1813 the "Analytical Society" was formed at Cambridge in England, which aimed to encourage in Britain the vigorous study of French higher mathematics. The influence of this movement reached the United States. In about ten years American teachers began to adopt French texts. Collateral events at West Point had the same tendency. There elementary mathematics was taught from 1808 to 1810 by F. R. Hassler, who was a graduate of the University of Berne in Switzerland. In 1817 Crozet, of the Polytechnic School in Paris, introduced descriptive geometry into West Point.

In 1819, John Farrar, of Harvard, brought out a translation of Legendre's Geometry, which, with translations made by him of other French and Swiss texts on mathematics, were at once widely adopted in the leading American colleges. American teachers were willing to turn to the French, not only for works on the calculus and celestial mechanics, but also for books on elementary mathematics. So it came about that Euclid was replaced by Legendre. In 1828 Charles Davies, professor at West Point, brought out an edition of Brewster's translation of Legendre's Geometry. Davies did not enunciate propositions with reference to and by the aid of the particular diagram used for the demonstration, and to that extent returned to the method of Euclid. Davies' edition became widely popular under the name of "Davies-Legendre," and was much used in the United States as late as the 70's.

Nature, Vol. 80, 1909, May 27, p. 374.

One of the earliest American geometries worthy of note, was that of Benjamin Peirce. The Harvard catalogue of 1838 announces that Freshmen take Peirce's Geometry. Peirce favored the use of infinitesimals and also the use of the term direction, a concept probably first used in this country by a Harvard teacher named Hayward in his geometry of 1829. Peirce's text did not become widely popular, for, like his other elementary books, it was too condensed for immature students. In 1843 or 1844, Harvard first made geometry a requirement for admission to College.

In 1851, Professor Elias Loomis, of Yale, issued a geometry which was revised in 1871. Loomis came under French influences as a student in Paris. In the second edition of his text he says: "The present volume follows substantially the order of Blanchet's Legendre, while the form of the demonstration is modeled after the more logical method of Euclid." It has been said of American writers, that while they have given up Euclid, they have modified Legendre's Geometry so as to make it resemble Euclid as much as possible. This applies to Loomis with greater force perhaps than to any other author.

In 1871 Professor Olney, of the University of Michigan, published a Geometry under two main heads:

I. *Special or Elementary Geometry,* comprising (1) Empirical Geometry, (2) Demonstrative Geometry, (3) Original Exercises in the Application of Algebra to Geometry, (4) Trigonometry.

II. *General Geometry* (Plane Loci).

Olney was a self educated man. He was a great teacher and had original ideas about teaching. It is said that he was prevented by his publishers from departing very far from the traditional classification. His ideas were novel and forecasted in many ways the present tendencies in mathematical teaching. His geometry shows that he attempted to correlate the various mathematical topics and to introduce applications to every day affairs. Olney's books were used quite extensively in the Middle West, but acquired no firm foothold in the East.

Just before the death of William Chauvenet, in 1870, appeared his *Geometry,* the only elementary book he wrote. Closely following French models, exhibiting a wonderful ease and grace of style, Chauvenet produced a remarkable book, which was used in many of the best schools. He included as a part of the work, an introduction to modern geometry. Perhaps no work on geometry ever published in the United States has been so highly respected as this.

In 1878 appeared the geometry of G. A. Wentworth, which is still in use. We omit all discussion of it, as also of later books which have been published in this country.

The researches on non-Euclidean Geometry, begun in the eighteenth century in Italy and Germany, and brought to fruition in the early part of the nineteenth century, did not produce appreciable effect upon the teaching

of elementary geometry until the last quarter of the nineteenth century. It was in 1867 and 1868 that Baltzer, Battaglini, Grunert and Hoüel brought Bolyai and Lobatchevsky to the attention of the mathematical public at large.

The new ideas have not affected the teaching of elementary geometry except in some of the definitions and postulates. They have assisted in the rejection of the definitions, "parallel lines are lines everywhere equally distant," and "parallel lines are straight lines which have the same direction." They have shown the futility of "proving" the parallel-postulate and have led to the use of the word "axiom," not as a "self-evident truth," but as a synonym for "postulate."

In conclusion, we note that, with the beginning of the twentieth century, England began once more to influence the teaching of geometry in the United States, through the so-called "Perry movement," and that Germany, which at no time during the nineteenth century affected geometrical teaching in America, makes itself felt at the present time through the pupils of Klein and Hilbert and through the international movement towards reform in the teaching of mathematics, headed by Klein.

ON THE SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS.

By G. A. MILLER, University of Illinois.

In the June-July number of this journal we proved the theorem that the necessary and sufficient condition that a given unknown has the same value in every solution of a consistent system of linear equations is that the rank of the matrix is reduced by omitting the coefficients of this unknown from this matrix. The following proof of this important theorem is probably more satisfactory to many readers. Suppose that the omission of the coefficients of x_1 reduces the rank of the matrix of the following consistent system of m equations in n unknowns:

$$\text{(A)}\qquad \begin{array}{l} a_{11}\,x_1+a_{12}\,x_2+\ldots+a_{1n}\,x_n+k_1=0,\\ a_{21}\,x_1+a_{22}\,x_2+\ldots+a_{2n}\,x_n+k_2=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{m1}\,x_1+a_{m2}\,x_2+\ldots+a_{mn}\,x_n+k_n=0. \end{array}$$

That is, if r is the rank of the matrix of this system, and if we let

$$a_{i2}\,x_2+a_{i3}\,x_3+\ldots+a_{in}\,x_n=f_r, \quad i=1,\,2,\,\ldots,\,m,$$

it is possible to find r constants, at least one being different from zero, so that

$$c_1\,f_1+c_2\,f_2+\ldots+c_r\,f_r\equiv 0.$$

We may assume that the first r equations of (A) are consistent but not linearly dependent and hence the coefficient of x_1 in the linear binomial equation

$$(c_1\;a_{11}+c_2\;a_{21}+\ldots+c_r\;a_{r1})x_1+c_1\;k_1+c_2\;k_2+\ldots+c_r\;k_r=0$$

is not zero. This implies that this binomial equation has one and only one solution. That is, x_1 has one and only one value no matter what values of the other unknowns may satisfy the given system of m equations. It may be desirable to mention here another theorem which is, however, more evident. This theorem may be stated as follows:

The necessary and sufficient condition that a given unknown can have the value zero in a consistent system of linear equations is that the matrix of the augmented system must be reduced by omitting the coefficients of this unknown from the system whenever the matrix of the system is reduced by this omission.

In fact, if the omission of these co-efficients does not reduce the rank of the matrix of the system, the unknown in question can have an arbitrary value and hence it can have the value zero. If, on the other hand, this omission does reduce the rank of the matrix as well as that of the augmented matrix, the system will remain consistent, and this unknown can have the value of zero. From the preceding theorem it follows that it can have no other value in this case, and also that it cannot have the value zero when the given omission reduces the rank of the matrix without also reducing the rank of the augmented matrix. That latter fact was suggested to me by Mr. G. Rutledge.

NOTES AND NEWS.

At the University of Chicago Dr. L. E. Dickson has been promoted to a full professorship in mathematics, and Dr. A. C. Lunn to an assistant professorship in applied mathematics. Also Dr. E. T. Wylczynski, of the University of Illinois, has been appointed to an associate professorship in mathematics.

At Columbia University Professor C. J. Keyser has been promoted to the headship of the department of mathematics and three new members of the staff have been appointed, namely Dr. H. E. Hawkes of Yale University and Dr. W. B. Fite of Cornell University to full professorships, and Dr. N. J. Lennes of the Massachusetts Institute of Technology to an instructorship.

At the University of Illinois Dr. J. B. Shaw and Dr. Arnold Emch have been appointed to assistant professorships in mathematics. The former comes from James Millikin University and the latter from the Cantonal College, Solothurn, Switzerland.

The Summer meeting of the American Mathematical Society was held at Columbia University on September 6, 7. About thirty members of the society were in attendance.

The Winter meeting of the Chicago Section of the Society will be held in Minneapolis, Minn., during the holiday week, in connection with the convocation of the American Association for the Advancement of Science.

The next annual meeting of the Central Association of Science and Mathematics teachers will be held in Cleveland, Ohio, on Friday and Saturday, November 25, 26. This year special emphasis will be given to mathematics and Professor David Eugene Smith of Teachers College, Columbia University, will deliver the chief address.

At the University of North Carolina, Professor Archibald Henderson is on leave of absence for the year and Mr. Guy M. Clements is associate professor in charge of the department of mathematics. Mr. Clements has done graduate work at Chicago and Harvard and was formerly instructor at Williams College.

The next annual conference of high schools in the State of Illinois will be held in the State University in Urbana on November 17, 18, and 19, 1910. The mathematics section will discuss the final report of the committee on a geometry syllabus which was appointed two years ago and made a preliminary report last year.

Mr. Walter W. Hart, who for several years has been head of the department of mathematics at the Shortridge High School in Indianapolis, has accepted a position in the department of education at the University of Wisconsin in connection with which he will have special relation to the work in mathematics in the high schools of the state.

At Spartanburg, S. C., Mr. Thomas M. Simpson is professor of mathematics in Converse College for Women and Mr. J. H. Peebles is professor of applied mathematics at Wofford College for men. The former is a graduate of the University of Virginia and the latter of Union College, Schenectady, N. Y.

The annual conference of secondary schools in co-operation with the University of Chicago will be at the University on Friday and Saturday, November 11, 12. The mathematics section will discuss various preliminary reports of committees acting under the International Commission.

Professor R. P. Baker, of the University of Iowa, completed the requirements for the Doctorate in mathematics at the University of Chicago during the past summer and received the degree at the September convocation.

At the September convocation, the University of Chicago conferred the doctor's degree upon Mr. W. H. Bates, instructor in mathematics at Purdue University, and upon Mr. A. D. Pitcher and Miss M. B. White, assistant professors of mathematics at the University of Kansas.

Miss Elizabeth R. Bennett, who received the doctorate at the University of Illinois last June, is instructor in mathematics at the University of Nebraska.

S. Lefsehetz has been given a Fellowship in mathematics in Clark University and is now engaged in study in that institution.

Our readers will be grieved to learn that on Oct. 7th, our valued contributer, and loyal friend of the *Monthly*, G. B. M. Zerr, died after an operation for pleuro-pneumonia. Many of our readers had learned to value him very highly. He solved more hard problems than any man in America so far as we know, and it was seldom that he passed a problem by in the *Monthly* in all the years of its existance. He has gone. Who will take his place, and help us in the difficult work he has so ably done for many years?

BOOKS AND PERIODICALS.

Projective Geometry. By Oswald Veblen, Preceptor in Mathematics, Princeton University, and J. W. Young, Assistant Professor of Mathematics in the University of Illinois. 8vo. Cloth, x+352 pages. Illustrated. Price, $4.00. Boston: Ginn & Co.

The authors of this work need no introduction to the mathematical public of America. They have in this work set forth the principles of Projective Geometry in an eminently scientific manner. Some of the main features of the text are "the view of geometry as a sequence of propositions deduced from explicitly stated assumptions, the free use of analytic methods on a purely synthetic basis, the role of groups and invariants in geometry, the distinction between projective and metric theorems on the basis of the group concept, the consideration of the complex as well as the real elements in synthetic arguments, and the geometry associated with an arbitrary field." The book contains a large number of exercises and illustrative examples.

It is the purpose of the authors to follow this volumn with a second, thus giving the teacher and student of mathematics the most complete treatment of Projective Geometry that has thus far been published in America.

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MATHEMATICAL PAPERS.

SOLUTIONS OF PROBLEMS—ALGEBRA.

GEOMETRY.

CALCULUS.

MECHANICS.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

THE AMERICAN MATHEMATICAL MONTHLY.

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MATHEMATICS IN PORTUGAL.

By G. A. MILLER, University of Illinois.

The fact that Portugal has so recently joined the list of republics increases interest at this time in the intellectual development of this little country with such a turbulent history. Her maritime eminence several centuries ago implies the use of mathematics at an early date and awakens the hope that we may find here independent developments of unusual value. A people that before the close of the fifteenth century found a route to India by sailing around the southern part of Africa must have possessed at an early date the basic elements of mathematics and astronomy. While the mathematical history of Portugal does not contain names that can be classed with Euler, Lagrange, Gauss, or Cauchy, yet it presents some names with which the mathematician is familiar. As such a name we may mention that of Nonius, who discovered before the middle of the sixteenth century, the fundamental properties of the important curves called loxodromes, which are important in navigation.*

Under the title ''Les Mathématique en Portugal'' M. Rodolphe Guimareas published in 1909 a second edition of a work which was originally prepared for the Universal Exposition of Paris in 1900. Nearly a hundred pages of this second edition are devoted to a sketch of the development of mathematics in Portugal. This is followed by a list of books and articles published by Portuguese writers together with occasional brief sketches as regards content or history. This list covers about 500 pages and constitutes a very useful aid towards obtaining a knowledge of the mathematical advances in this country.

From this list and from the historical sketch which precedes it, one can readily see that the Portuguese have been more successful in applying the results discovered elsewhere than in making important advances in pure mathematics. The professors in their foremost educational institution, the University of Coimbra, founded at Lisbon in 1290, and finally transferred to Coimbra in 1557, after having been moved to and fro several times, gained

*Cf. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. 2 (1900), p. 390.

distinction by their ability to assimilate and adapt knowledge to local conditions and needs, rather than by making extensive additions. These general conditions help to explain the universally acceded mathematical pre-eminence of F. Gomes Teixeira, whose works are being published by the Portuguese government. Two recent volumes of these works may be of sufficient general interest to merit a brief description here.

These two volumes bear respectively the dates 1908 and 1909, and constitute volumes 4 and 5 of the collected works of Teixeira. They are devoted to a study of the properties of the remarkable curves which have received special names. Volume 4 is devoted to algebraic curves, and covers about 400 large pages, while volume 5 is devoted to transcendental curves, and covers about 500 pages. These volumes are largely a French translation of the Spanish work by the same author, entitled "Tratado de las curvas especiales notables," which, together with the well known work of Loria, was crowned by the Academy of Sciences of Madrid in 1899. The subject proposed by this Academy read as follows: "A methodical catalogue of all the curves of any class which have received a special name, with a succinct idea of the form, equations and the general properties of each of them, and a reference to the works or authors who first mentioned them." An extensive and meritorious work devoted to such a broad subject has evidently a wide field of usefulness, and should be of unusual interest, even to those who do not hope to penetrate deeply into mathematics, since special effort has been made to present the results in a very elementary manner. The French edition differs from the older Spanish edition in regard to the amount of space devoted to each curve, this amount being much larger, as a rule, in the former of these editions. Several curves have also been added, and as the French language is more commonly known among scholars than the Spanish, it may be assumed that the new edition will have a wider field of usefulness than the older one enjoyed. The number of distinct curves treated is about two hundred, so that the average amount of space devoted to one curve is not large. The conics were excluded, since their fundamental properties are so well known.

In the preface, the author states that he studied the form, construction, rectification and quadrature, the properties and the history of each curve. He considered the relations of each curve to the others, and indicated the problems which led to a study of them, giving references to the authors whenever this was possible. Each volume is provided with a list of the curves studied, a list of authors mentioned and a Table of Contents, so that it offers a convenient work of reference. It was printed at the University Press of Coimbra and published by order of the Portuguese government.

With respect to mathematical journals, Portugal has also a very respectable record, if we bear in mind that her area and population are both much less than those of the State of Pennsylvania. The two journals which

are most favorably known among mathematicians were founded by Teixeira. The older of these, entitled *Jornal de sciencias mathematicas e astronomicas,* was founded in 1877. It was superceded in 1905 by the *Annals Scientificos da Academia Polytechnica do Porto,* which is not restricted to mathematics, but has thus far devoted considerable space to this subject. The second number of the current volume begins with an article by P. Appell on the deduction of the polynomials of Hermite from those of Legendre.

From what precedes it is evident that the youngest sister republic cannot be classed with the foremost mathematical countries of the world, but it is equally true that, if we consider her size and population, she has made a very respectable record and is doing so at the present time. It is to be hoped that the new form of government will tend to elevate the educational opportunities of the masses and to put new life also into the higher institutions. In the sixteenth century Lisbon was one of the intellectual centers of Europe, and the later scientific achievements under adverse conditions inspire the hope that with the improvement of these conditions there may come a return of intellectual eminence. Even at the present time some of the Portuguese literature has decided value, both for the investigator and also for those who seek general mathematical knowledge.

ON THE NEW COURSE IN MATHEMATICS IN THE JAPANESE NORMAL SCHOOLS.

By YOSHIO MIKAMI, Phara in Kazusa, Japan.

The Department of Education of the Japanese government recently issued a new course in mathematics for the normal schools; that is, for the schools where the teachers of the primary schools are educated. The courses for men students are not the same as those for women. We begin by giving a brief account of the former.

The course consists of two parts, requiring one and four years respectively. In the former of these (the preliminary course) six hours per week are devoted to mathematics, while the number of hours devoted to this subject during each week of the remaining four years are respectively four, three, three, and two. Hence the total number of hours devoted to mathematics in these normal schools is somewhat less than that of the middle schools.

The six hours of mathematics in the preliminary course are devoted to arithmetic, such as is taught in higher primary schools.*

*In Japan primary schools have two courses, ordinary and higher. The ordinary course lasts six years and the other two years. Children are admitted to the primary school when they reach their sixth year.

In the main course, considerable importance is attached to the practical side of mathematical instruction, emphasizing the connection between various branches of it as much as possible. In this respect the plan differs widely from the prevailing course for the middle school mathematics, where no connection between even arithmetic and algebra is attempted. The new plan is certainly a step towards improvement.

In the first year arithmetic and algebra are united under a common heading, and the subjects taught are integral numbers, the four fundamental operations, decimals, common fractions, negative numbers, integral expressions, linear equations. Geometry is also taught, its theme subjects consisting of the angle, parallels, triangles, and paralellograms, including the areas of rectilinear figures, and the circle. With geometry arithmetical calculations are associated.

In the second year arithmetic, algebra and geometry are all taught under a common heading. The subjects are fractional expressions and fractional equations, square and cube roots, quadratic equations, irrational equations, proportion, similar figures and areas.

In the third year the trigonometric functions of acute angles, together with the solution of triangles, arithmetic and geometric progressions and problems in interest are taught. Besides these, lessons on book-keeping and on the teaching of arithmetic for primary schools are given during the year.

In the fourth year algebra disappears and solid geometry alone is taught. Planes, solid angles, prisms, pyramids, circular cylinders, circular cones, spheres, and the volumes of these solids, are the subjects now considered. Arithmetical mensuration is particularly recommended to be associated with the subjects of solid geometry.

For the women students the preliminary course and the first and third years of the main course are provided each with one hour less than for the men. But all the subjects for the latter are preserved for the women. The only difference is that geometry is begun in the second year, the trigonometric functions are omitted, and that some parts are arranged differently. It will, however, be understood that the subject matter is considerably simpler than that for the men, since the women receive a smaller number of hours of instruction.

Besides the main course of four years, there are established simpler courses of two years and of one year. In the first year of the two years course mathematics is taught four hours a week, and in the second year three hours. Here the subject matter of the first and second years of the four year course is taught in the first year, and that of the third and fourth years in the second year. But it is evident that some parts are necessarily simplified, owing to the shorter time.

In the one year course all the main subjects of the four years of the main course are taught, only with simplifications and omissions.

In addition to the subjects mentioned above, mental arithmetic, the abacus arithmetic with the use of the *soroban*, and the geometrical loci and problems of construction are taught at convenient times. The *soroban* is an abacus that was introduced from China. Its introduction is believed by some writers to have been about 1600, but it may have been made at a far earlier period. However, it was since the middle of the seventeenth century that it was popularized in Japan. Although there was also another sort of abacus, the *sangi*, or calculating pieces, the sole help in daily use for calculations was the *soroban*; for the *sangi* were too cumbersome and better adapted to more complicated calculations. Even since the introduction of the Occidental style of calculation in the middle part of the last century, the *soroban* has not entirely disappeared, and it is still widely used. This is the reason why the *soroban* arithmetic is taught at present in primary schools, together with the Occidental arithmetic.

The adoption of the practical side of mathematical teaching in normal schools will certainly be against the wishes of those who insist that beginners should have theoretical instruction only. But this plan appears to prove successful in training young minds to the assimilation of mathematical ideas; it is especially in agreement with the development of the Japanese character which always looks towards the practical.

It is understood that the new course is not free from faults, but it is recommended to the teacher as a standard, and it is believed that the skill and the power of adaptation already displayed by the Japanese in so many directions will enable them to improve upon it and to develop eventually a still more complete and satisfactory plan. It is worthy of note that Japan is engaged in this development at the same time that all the peoples in the civilized world seem to be considering the possible improvements in the teaching of mathematics.

ON A MEAN DIFFERENCE PROBLEM THAT OCCURS IN STATISTICS.

By H. L. RIETZ, University of Illinois.

1. *Introduction.* In making a comparison of the differences between the highest and lowest examination marks of a pupil in a given subject, with the corresponding difference for the same pupil in some other subjects, E. G. Dexter dealt with data such that, in one subject, say in mathematics, each pupil had ten distinct marks, while in another subject, say in Latin, each pupil had only two or three such marks. In this case, it seems reasonable to expect that, other things being equal, the extreme marks in mathematics would tend to differ more that the extreme marks in Latin.

In considering the question of comparing the average values of the differences between these extreme marks, for a large number of pupils, Dexter proposed to me the following problem:

An urn contains 101 counters marked, 0, 1, 2, ..., 100. Drawings are made at random taking r at a time, always replacing the counters before drawing again. Find the mean difference between numbers on the counters taken two at a time, and the mean difference between the most extreme numbers on counters taken 3, 4, 5, ..., r at a time.

I present the solution here not merely because it seems to be a special problem of some interest, but mainly because of its relation to a general problem in statistics, known as Galton's Difference Problem.*

It may be worth remarking here that the solution of Galton's Difference Problem offers, as a special result, an answer to the question of the most suitable ratio between the values of first and second prizes in a competition.

2. To solve the problem proposed by Dexter, let us define the mean difference as the sum of the products of each separate difference by the probability of its occurrence.

a) The case of drawing two counters at a time.

We may form a difference of 100 in one way, of 99 in two ways, of 98 in three ways, and so on. Hence the mean value is

$$M=\frac{100.1+99.2+\ldots+1.100}{1+2+..,+100}=\frac{\sum_{1}^{100} x(101-x)}{\sum_{1}^{100} x}=34.$$

b) The case of drawing three counters at a time.

We may form a difference of 100 between extremes in 1×99 ways, of 99 in 2×98 ways, of 98 in 3×97 ways, and so on. Hence, the mean difference between extremes is

$$M=\frac{100.99.1+99.98.2+\ldots+2.1.99}{99.1+98.2+\ldots+1.99}=\frac{\sum_{1}^{99} x(101-x)(100-x)}{\sum_{1}^{99} x(100-x)}=51.$$

c) The case of drawing r counters at a time.

We may form a difference of 100 in $1\times{}_{99}C_{r-2}$ ways, of 99 in $2\times{}_{98}C_{r-2}$, of 98 in $3\times{}_{97}C_{r-2}$ ways, and so on. From this, we find, by a very slight reduction, that the mean difference between extreme numbers on the r counters is given by

*Pearson, *Biometrika,* Vol. I, pp. 390-399.

$$M = \frac{\sum_{1}^{102-r} (101-x)x(100-x)(99-x)\ldots(103-r-x)}{\sum_{1}^{102-r} x(100-x)(99-x)\ldots(103-r-x)}$$

To evaluate this expression for any special value of r requires only the sum of series that are integral powers of the natural numbers; that is, of the series $1^s+2^s+3^s+\ldots+n^s$.

If, corresponding to each mark from 0 to 100, there are a fixed number t of counters in the urn instead of one only, the problem is solved by a slight extension of the above; since, when $t \geqq r$, there are a certain number of ways in which r counters with equal numbers may be drawn.

On the side of statistical applications, obviously data arranged in frequency groups with respect to some character rarely even approximate to constant values at equal intervals as suggested by the counters in the above problem. For example, examination marks on a percentage basis are, in general, much more frequent at some intervals of the range from 0 to 100 than at others. Thus, as an illustration, the examination marks of a certain group of 1255 students in foreign languages, may be exhibited in frequency groups as follows:

Intervals:	50-52.5	52.5-57.5	57.5-62.5	62.5-67.5	67.5-72.5
Frequency:	1	1	2	6	20

72.5-77.5	77.5-82.5	82.5-87.5	87.5-92.5	92.5-97.5	97.5-
89	195	283	323	307	28

This arrangement of a totality with respect to some character is called a *frequency distribution.* It is a reasonable assumption that the frequency of occurrence follows some law of probability; and, to make the matter fairly simple, we might perhaps assume an ideal system of grading such that the frequency of marks from 0 to 100 should be proportional to the 101 terms in the expansion of $(p+q)^{100}$ where $p+q=1$. But the problem of the mean difference between extreme numbers in drawing numbered counters with such a law of distribution becomes very complicated. On this account, we seek an analytic treatment that will bring in functions to which we can apply the calculus in performing summations.

3. *Continuous treatment of mean differences.* The problem proposed by Dexter has a continuous analogue in the following:

On a line segment AB of length l, r points $x_1, x_2, \ldots, x_r$, are selected at random. Find the mean value of $|x_r-x_1|$ where x_1 and x_r denote the extremes of the r points.

For the simple case of two points, this problem is solved in some standard books* on the theory of probability.

It is assumed for this problem that the probability of a point selected belonging to a constant interval ∂x of the line AB is the same no matter where the interval is chosen.

Let x' be the abscissa of any point on l and x that of any point, where $x>x'$. Then the probability of a point first selected belonging to an interval $\partial x'$, the second selected to ∂x, and the remaining $r-2$ to the interval between x' and x is

$$\frac{\partial x'}{l}\cdot\frac{\partial x}{l}\cdot\left(\frac{x-x'}{l}\right)^{r-2}.$$

But there are $r(r-1)$ ways to select a group of 1, 1 and $r-2$ out of r things. Hence, we have for the mean difference

$$r(r-1)\int_0^l\int_0^x\frac{\partial x'}{l}\cdot\frac{\partial x}{l}\left(\frac{x-x'}{l}\right)^{r-2}(x-x')=\frac{r-1}{r+1}l.$$

This result gives, for a constant and continuous distribution from 0 to 100, a mean difference of $33\frac{1}{3}$, when selected in sets of two to compare with 34 for a distribution of a single individual at intervals of one unit, and a mean difference of 50 to compare with 51 when selections are made three at a time.

4. *Galton's Difference Problem.* Let us consider a class of objects and let some numerical mark be attached to the individuals of this class. For example, we may think of statistical data representing the statures of men of a certain class, or the examination marks of a group of pupils. All that is essential is a totality each individual of which is marked by some number. Let the marks be represented as abscissas of points on the x-axis, and let us assume that a function $y=f(x)$ exists such that $y\partial x$ gives the probability that a point, located by the mark of an individual taken at random from the totality, belongs to the interval from x to $x+\partial x$. If a sample of n be selected at random out of the totality, we set the problem of finding the mean difference between the pth and $(p+1)$th individuals of the sample, when the n individuals of the sample are arranged in order of magnitude with respect to the character.

This is Galton's Difference Problem, and we need only add such differences from $p=1$ to $p=n-1$ to solve the problem of the mean difference between extremes in samples of n.

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*Czuber, *Wahrscheinlichkeitsrechnung*, 1903, p. 76.
Borel, *Eléments de la Théorie des Probabilités*, p. 97.

Pearson stated Galton's Problem in its general form, and obtained* for the mean difference between the pth and $(p+1)$th individual

$$D_p=\frac{n!}{(n-p)!\ p!}\int_{-\infty}^{+\infty}\alpha^{n-p}(1-\alpha)^p d\alpha \ldots \ (1),$$

where
$$\alpha=\int_{-\infty}^{x} f(x)\,dx \ \ldots \ (2).$$

The problem is thus reduced to one of quadrature. Suppose $f(x)$ is the probability function of Gauss, written in the form $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$; then, tables of values of α with the argument x are easily accessible. In this case, by quadration, from (1), using Simpson's two-thirds rule and intervals of 0.2σ, I have evaluated D_p for $n=2$ and $n=10$. The values for $n=3$ are given by Pearson. The results are:

For $n=2$, $D_1=1.13\sigma$,†
$n=3$, $D_1=0.846\sigma$, $D_2=0.846\sigma$,
$n=10$, $D_1=0.540\sigma$, $D_2=0.346\sigma$, $D_3=0.282\sigma$, $D_4=0.253$,
$D_5=0.247$, $D_6=0.253$, $D_7=0.282\sigma$, $D_8=0.346\sigma$, $D_9=0.540\sigma$.

Let E_n=the mean difference of extreme values when samples of n are taken, then

$$\begin{aligned} E_2&=1.13\sigma,\\ E_3&=1.69\sigma, \ \ldots \ (3).\\ E_{10}&=3.09\sigma. \end{aligned}$$

To indicate the significance of these results, we may say that in selecting at random a sample of 10 from a Gaussian distribution, we have a mean difference between extremes of $\frac{3.09}{1.13}=2.73$ times as much as when we take only two at a time, and $\frac{3.08}{1.69}=1.82$ times as much as when we take three at a time.

To apply these results to a reasonable distribution for examination marks, suppose that marks range from 50 to 100, and that the frequencies of occurrence of given marks 50, 51, 52, ... are proportional to the terms in

*Loc. cit., p. 392.
*The last figure in these calculations may be of doubtful value.

the expansion of $(\frac{1}{2}+\frac{1}{2})^{50}$. Then the distribution is well described by a Gaussian function in which

$$\sigma=\sqrt{(50\times\tfrac{1}{2}\times\tfrac{1}{2})}=\frac{5\sqrt{2}}{2}.$$

In this case,

$$E_2=3.99,$$
$$E_3=5.97,$$
$$E_{10}=10.92.$$

The results (3) give a precise notion as to the values of the mean difference between extreme individuals of a small sample taken at random from a totality that is distributed in accord with Gauss's law.

Since many frequency distributions not well described by Gauss's law are well described by its generalizations, it seems likely that results, such as concern us here, when derived from Gauss's curve apply at least roughly to many more general types of frequency distributions that occur in statistics. To illustrate, the distribution of examination marks given on p. 237 is not at all well fitted by numbers proportional to the terms in the expansion of $(\frac{1}{2}+\frac{1}{2})^{50}$; but, if we use the ungraduated values there given to evaluate (1) by quadrature, we obtain, for the mean difference in taking sets of two, the value 7.71. But, if we compute σ as the square root of the mean square of the deviations of observations from their mean value, the result is 7.42. Then, from (3), $E_2=8.38$ to compare with 7.71; and, hence, the result from the assumption of the law of Gauss gives at least a good general notion of the mean difference in question.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

342. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Prove that $\frac{1}{1.2.3.4}+\frac{1}{5.6.7.8}+\ldots=\frac{1}{4}\log 2-\frac{1}{24}\pi$. [Hobson's *Plane Trigonometry*, page 348.]

Solution by V. M. SPUNAR, Cleveland, Ohio, and the PROPOSER.

The general term is

$$\frac{1}{(4n-3)(4n-2)(4n-1)4n}=\tfrac{1}{6}\left(\frac{1}{4n-3}-\frac{3}{4n-2}+\frac{3}{4n-1}-\frac{1}{4n}\right)$$

$$=\tfrac{1}{6}\left(\frac{2}{4n-3}-\frac{2}{4n-2}+\frac{2}{4n-1}-\frac{2}{4n}\right)-\tfrac{1}{6}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)-\tfrac{1}{6}\left(\frac{1}{4n-2}-\frac{1}{4n}\right).$$

Therefore the series may be written

$$\tfrac{1}{3}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots)-\tfrac{1}{6}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots)-\tfrac{1}{12}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots)$$

$$=\tfrac{1}{4}\log 2-\tfrac{1}{6}\tan^{-1}1=\tfrac{1}{4}\log 2-\tfrac{1}{24}\pi.$$

343. Proposed by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

A, on contracting to execute a piece of work for \$300 and finding after working alone one day that he had finished but 1% of the entire work, engaged B to assist him at the beginning of the second day, with the understanding, that B on each day was to do 6% as much work as had been completed previously, while A each day was to do an amount of work equal to 1% of the unfinished work at the close of the day before. At the completion of all the work the \$300 were divided between A and B in proportion to the amount of the work each had performed.

Required—(1) The number of days to do the work; (2) on which day would the daily earnings of A and B be the same; and (3) the amount of money each was paid under the agreement.

Solution by the PROPOSER.

Let $r=1\%$ and $r_1=6\%$; and let x, a variable, $=$the time in days to do the whole work, or 1; and let the whole work completed to the end of the days 0, 1, 2, 3, ..., x, $x+1$, ..., be represented by the functions u_0, u_1, u_2, ..., u_x, u_{x+1}, ... The whole work completed to the end of the day $x+1$, or u_{x+1}, is equal to the work completed to the end of x days, or u_x, plus the part completed by A on the day x, or $r(1-u_x)$, plus the part completed by B on the day x, or r_1u_x. Equate the functions and have:

$$u_{x+1}=u_x+r(1-u_x)+r_1u_x\dots(1);$$
$$\text{or, } u_{x+1}-(1-r+r_1)u_x=r\dots(2).$$

Give the equation numerical values and we have

$$u_{x+1}-(1.05)u_x=0.01\dots(3).$$

Equation (3) belongs to the Calculus of Finite Differences. Integrate it and have

$$u_x - C(1.05)^x = -0.2 \ldots (4).$$

Equation (4) is true for all values of x, and is therefore true when $x=0$. When $x=0$, $C=0.2$; and this value of C gives

$$u_x = 0.2[(1.05)^x - 1] \ldots (5).$$

To find (1) the time to complete the work. When the work is completed $u_x = 1$. Substitute this value of u_x in (5) and have, $(1.05)^x = 6$; or $x \log(1.05) = \log 6$; or $x = \log 6/\log(1.05) = 36.72+$days.

To find (2), when the earnings of each are the same, in equation (1) transfer u_x to the first member and have

$$u_{x+1} - u_x = r(1-u_x) + r_1 u_x \ldots (6).$$

As u_{x+1} is the whole work completed in $x+1$ days, and u_x is the whole work completed in x days, their difference is the work completed on the day $x+1$. Substitute in the second member of (6), the value of u_x from (5) and have

$$u_{x+1} - u_x = 0.002[6-(1.05)^x] + 0.012[(1.05)^x - 1] \ldots (7).$$

In (7), for $x+1$ write x, as x is a variable, and have

$$u_x - u_{x-1} = 0.002[6-(1.05)^{x-1}] + 0.012[(1.05)^{x-1} - 1] \ldots (8).$$

The first member of (8) is the work completed in x days, and the two terms of the second member show the work completed by A and B, respectively, on the day x, and as these two terms, under the conditions of the problem, must be equal, equate them, reduce, and have

$$6-(1.05)^{x-1} = 6[(1.05)^{x-1} - 1] \ldots (9);$$

or $7(1.05)^{x-1} = 12$; or $(1.05)^{x-1} = 12 \div 7$; or $(x-1)\log(1.05) = \log(12 \div 7)$;
or $x = 1 + \log(12 \div 7)/\log(1.05) = 12.05$ days, or 12 days.

To divide the money (3) recur to equation (8) and observe that the exponents of the two terms in the second member are one degree lower than

the subscript x in u_x. This is a general law and it enables us to generate the 36 equations of work completed as x takes different values from 1 to 36.

End of 1 day, $u_1-u_0=0.002[6-(1.05)^1]+[0]\ldots(10)$;
End or 2 days, $u_2-u_1=0.202[6-(1.05)^1]+0.012[(1.05)^1-1]\ldots(11)$;

. . . .

End of 35 days, $u_{35}-u_{34}=0.002[6-(1.05)^{34}]+0.012[(1.05)^{34}-1]\ldots(44)$; and
End of 36 days, $u_{36}-u_{35}=0.002[6-(1.05)^{35}]+0.012[(1.05)^{35}-1]\ldots(45)$.

Add the equations and have

$$u_{36}-u_0=0.002\{216-[(1.05)^0+(1.05)^1+\ldots+(1.05)^{35}]\}$$
$$+0.012[(1.05)^1+(1.05)^2+\ldots(1.05)^{35}-35]\ldots(46).$$

Sum the first term of the second member of (46) for the work completed by A in 36 days, and sum the second term for the work completed by B in 35 days, and have:

for A's work, $0.002\{216-20[(1.05)^{36}-1]\}=0.2403+$;
and for B's work, $0.012\{21[(1.05)^{35}-1]-35\}=0.7180+$.

A's work for 36 days+B's work for 35 days$=0.2403+0.7180=0.9583+$.

The unfinished work$=1-0.9583=0.0417-$; work to be finished by A and B in 0.72 day.

For A's unfinished part we have $(0.0417)(0.01)(0.72)=0.0003$; and $0.2403+0.0003=0.2406=$the total part completed by A.

For B's unfinished part we have $(0.8583)(0.06)(0.72)=0.0414$; and $0.7180+0.0414=0.7594$, the total part completed by B.

A's total+B's total$=0.2406+0.7594=1$, as it should.

A's share of the money, therefore, $=\$300\times(0.2406)=\$72.18$; and B's share$=\$300\times(0.7594)=\$227.82$.

Also solved by V. M. Spunar.

GEOMETRY.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

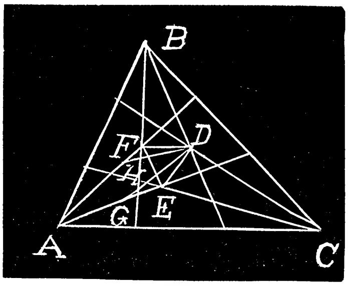
No satisfactory solution of this problem has been received.

370. Proposed by R. C. ARCHIBALD, Paris, France.

The trisectors of the angles of any triangle ABC are, in order, AF, AE, CE, CD, BD, BF. Show synthetically that D, E, F are the vertices of an equilateral triangle.

Solution by A. H. HOLMES, Bruntwick, Me.

Let ABC be the triangle with AE, AF trisectors of $\angle BAC$; BF, BD trisectors of $\angle ABC$, and CE, CD trisectors of $\angle ACB$. Draw DE, DF, and EF. Then DEF is an equilateral triangle. From D draw DG parallel to AF and cutting AE in G. Take, on AF, $AH=DG$. Then $\angle GDH=\angle EAF=\frac{1}{3}\angle BAC$.



Suppose the triangle GED to be moved so that GE is colinear with AF and the point E is at the point F. Then since DG is parallel to AF and $\angle EAF=\angle FAB$, DG will be parallel to AB.

$\therefore \triangle DEG$ and $\triangle AFB$ are similar, and $\angle EDG=\angle ABF=\frac{1}{3}\angle ABC$. Similarly, it may be shown that $\angle FDH$ is equal to $\frac{1}{3}\angle ACB$.

$\therefore \angle EDF=\frac{1}{3}(\angle ABC+\angle ACD+\angle BAC)=60^\circ$. In the same way, $\angle DEF$ is shown to be equal to 60°.

$\therefore \triangle DEF$ is an equilateral triangle.

371. Proposed by W. S. HUGHES, Student, Williams College.

A right circular cone is cut by two parallel planes, one passing through the vertex, and each cutting both nappes. Are the straight lines which constitute the first section parallel to the asymptotes of the hyperbola forming the other section?

Solution by FRANK LOXLEY GRIFFIN, Ph. D., Assistant Professor of Mathematics, Williams College.

As the X-axis take the axis of the cone, and as the Y-axis the intersection of the first cutting plane with the plane through the vertex perpendicular to the X-axis. Then the equation of the cone is $y^2+z^2=m^2x^2$, where m denotes the tangent of one-half the vertex angle of the cone. In the plane XOZ rotate the axes OX and OZ through an angle α, such that OX shall be in the first cutting plane. Then the cone and the cutting planes are given respectively by

(1) $$y^2+(x'\sin\alpha+z'\cos\alpha)^2=m^2(x'\cos\alpha-z'\sin\alpha),$$

(2), (3) $$z'=0, \qquad z'=d,$$

where d is the distance between the cutting planes. Now the straight lines of the first section are given by (2) and the equation obtained by making $z'=0$ in (1), say $B^2x'^2-y^2=0$. And the hyperbola is given by (3) and an

equation obtained from (1) for $z'=d$, whose second degree terms, being those of (1) not containing z', are the same: $B^2(x'-k)^2-y^2=l^2$. Thus the asymptotes $y=\pm B(x'-k)$, $z'=d$ are parallel to the first lines $y=\pm Bx'$, $z'=0$.

Also solved by S. G. Barton and V. M. Spunar.

MECHANICS.

249. Proposed by the late G. B. M. ZERR, Ph. D.

A load P is supported by three strings of equal size attached at the vertices of a triangle, sides a, b, c lying in a horizontal plane. The load is vertically under the centroid of the triangle at a distance h from it. Find the stresses in the strings.

Solution by the PROPOSER.

Let h_1, h_2, h_3 be the medians of the triangle; T_1, T_2, T_3 the stresses on the strings attached to A, B, C, respectively; D, the point where the load is fixed; G, the centroid; H, the mid-point of BC.

$\angle ADG=\theta$, $\angle BDG=\phi$, $\angle CDG=\psi$, $\angle BGH=\rho$, $\angle CGH=\mu$.

$$p=\cos\theta=3h/\sqrt{(9h^2+4h_1^2)}=3h/\sqrt{(9h^2+2b^2+2c^2-a^2)}.$$
$$q=\cos\phi=3h/\sqrt{(9h^2+2a^2+2c^2-b^2)}.$$
$$r=\cos\psi=3h/\sqrt{(9h^2+2a^2+2b^2-c^2)}.$$
$$m=\sin\phi\sin\rho=3ab\sin C/\sqrt{[(9h^2+2a^2+2c^2-b^2)(2b^2+2c^2-a^2)]}.$$
$$n=\sin\psi\sin\mu=3ab\sin C/\sqrt{[9h^2+2a^2+2b^2-c^2)(2b^2+2c^2-a^2)]}.$$

Let E=Young's modulus, β=sectional area of string, p_1=elongation AD, p_2=elongation BD, p_3=elongation CD. Then

$$P=pT_1+qT_2+rT_3\ldots(1),$$
$$mT_2=nT_3\ldots(2),$$
$$T_1p_1+T_2p_2+T_3p_3=\text{minimum}.$$

Now $p_1=\dfrac{AD.T_1}{E\beta}$, $p_2=\dfrac{BD.T_2}{E\beta}$, $p_3=\dfrac{CD.T_3}{E\beta}$.

$AD.p=BD.q=CD.r=h$.

Hence, $\dfrac{h}{E\beta}\left(\dfrac{T_1^2}{p}+\dfrac{T_2^2}{q}+\dfrac{T_3^2}{r}\right)=\text{minimum}\ldots(3)$.

From (1), (2) and (3) we get

$$pdT_1+qdT_2+rdT_3=0\ldots(4),$$
$$mdT_2=ndT_3\ldots(5),$$
$$T_1dT_1/p+T_2dT_2/q+T_3dT_3/r=0\ldots(6).$$

Eliminating dT_1, dT_2, dT_3 between (4), (5) and (6), we get

$$T_1(qn+rm)qr=T_2p^2m+T_3p^2qm\ldots(7).$$

From (1), (2) and (7) we get

$$T_1=\frac{Pp^2(rn^2+qm^2)}{p^3(rn^2+qm^2)+qr\ (qn+rm)}.$$

$$T_2=\frac{Pqrn(qn+rm)}{p^3(rn^2+qm^2)+qr\ (qn+rm)}.$$

$$T_3=\frac{Pqrm(qn+rm)}{p^3(rn^2+qm^2)+qr\ (qn+rm)}.$$

Errata.—Begin with problem 240. Mechanics, page 194, Vol. XVI, and number them consecutively through Vol. XVII. Problem 247 in the last issue should be 248. Note that 244 on page 48, Vol. XVII is the same as 241, page 21.

250. Proposed by C. N. SCHMALL, New York City.

A smooth circular table is surrounded by a smooth vertical rim. A ball of elasticity e is projected from a point at the rim in a line making an angle ϕ with the radius through that point. Show that the ball will return to the starting point after the second impact if

$$\tan\phi=\sqrt{\frac{e^3}{e^2+e+1}}.$$

Solution by the late G. B. M. ZERR, Ph. D.

Let A be the point of projection; B, C the points of first and second impact; O, the center of the table; $\angle OAB=\angle OBA=\phi$, $\angle OBC=\angle OCB=\theta$, $\angle OCA=\angle OAC=\psi$.

Then $\tan\phi=e\tan\theta$, $\tan\theta=e\tan\psi$.

Hence, $\tan\phi=e\tan\theta=e^2\tan\psi$. Also, $\phi+\theta+\psi=\frac{1}{2}\pi$.

$\therefore 1=\tan\phi\tan\theta+\tan\phi\tan\psi+\tan\theta\tan\psi$. Whence

$$1=\tan^2\phi\left(\frac{1}{e}+\frac{1}{e^2}+\frac{1}{e^3}\right).$$

Hence, $e^3=(1+e+e^2)\tan^2\phi$, from which we obtain,

$$\tan\phi=\sqrt{\frac{e^3}{e^2+e+1}}.$$

Solved in a similar manner by J. Scheffer.

PROBLEMS FOR SOLUTION.

GEOMETRY.

379. Proposed by G. I. HOPKINS, Manchester, N. H.

Construct the triangle, having given base, vertical angle, and ratio of its altitude to difference of other two sides.

380. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

$A\ B\ C\ D$ is a quadrilateral, sides in order a, b, c, d, and $B+D=\theta$. Express the diagonals in terms of a, b, c, d, θ.

MECHANICS.

355. Proposed by the late G. B. M. ZERR, Ph. D.

Assuming the resilience of volume of mercury to be constant at all depths and to be 54. 20×10^{10} in $C.\ G.\ S.$ units and that a mile = 160933 centimeters. Find the depth of an ocean of mercury at a point where its density is double the surface density, 13.596.

356. Proposed by the late G. B. M. ZERR, Ph. D.

A cantilever beam length a is loaded with c pounds per running foot at its fixed end and increases uniformly to b pounds per running foot at its free end. Find the deflection at the free end due to this load.

357. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

A portion of a circular cylinder cut off by two planes through the axis rests with its curved surface on two rough horizontal rails parallel to its axis, the coefficients of friction μ_1, μ_2 at upper and lower rails respectively. If the body is in limiting equilibrium at both rails when the plane through the axis and the center of gravity is perpendicular to both rails, find the distance of the center of gravity in terms of the distance betwen the rails, the inclination of their plane to the horizon, and the cofficients of friction.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

177. Proposed by J. EDWARD SANDERS, U. S. Weather Bureau Office, Chicago, Ill.

Factor (if possible) 1,111,111,111,111,111,111.

NOTES AND NEWS.

Dr. George Bruce Halsted has been elected Corresponding Member of the Société des Sciences Physiques et Naturelles de Bordeaux, in whose Mémoires appeared his work La contribution non euclidienne à la philosophie. F.

Messrs. Ginn and Company announce the publication of a book entitled *The Hindu-Arabic Numerals,* by David Eugene Smith, of Columbia University, and Louis C. Karpinski, of the University of Michigan. The work gives a complete story of the rise and development of the numerals, and is illustrated with numerous facsimiles from early inscriptions and manuscripts, most of which have not hitherto been published in connection with this subject, and all of which contribute to a very marked degree to an understanding of the problem. S.

The Mathematics Club of Columbia University is a new organization for the purpose of promoting personal and intellectual relations among the graduate students and faculty in the department of mathematics. Regular meetings of an informal character will be held for the reading and discussion of papers. This organization is in addition to the more formal Mathematical Colloquium of long standing in the department. A similar organization has existed at the University of Chicago for several years, to supplement the work of the Mathematical Club and Seminar and to form a rallying point for mathematical students early in their graduate course. S.

In the Central Association of Science and Mathematics Teachers during the past few years, important reports have been presented on the teaching of biology, chemistry, physics, algebra, geometry, and unified mathematics. In addition to these formal reports numerous symposiums and discussions have been published in *School Science and Mathematics.* At the recent annual meeting in Cleveland, there was presented an important report on the fundamental findings of all the previous documents, including both those common to the various subjects and those peculiar to special subjects. This report is well worth the careful study of all teachers of mathematics. It shows the constructive and coherent activity of this association. Copies may be obtained from the press of *School Science and Mathematics,* 2059 East 72nd Place, Chicago, Illinois. S.

BOOKS.

Text Book on Practical Astronomy. By George L. Hosmer, Assistant Professor of Civil Engineering, Massachusetts Institute of Technology. 8 vo. Cloth, vii+205 pages. 78 figures. Price, $2.00 net. New York: John Wiley & Sons.

The purpose of this volume, we are told, is to furnish a text in practical astronomy, especially adapted to the needs of civil engineering students who can devote but little time to the subject and who are not likely to take up the advanced study of astronomy. The text deals, therefore, largely with that class of observations which can be made with surveying instruments, the refinements of observations and the rigorous mathematical formulation of principles being left for the theoretical student to supply.

The work is in full agreement in its methods with the spirit of modern education, and will find a useful place in practical teaching. F.

An Elementary Treaties on the Dynamics of a Particle and of Rigid Bodies. By S. L. Loney, M. A., Professor of Mathematics at the Royal Holloway College (University of London), sometime Fellow of Sidney Sussex College, Cambridge. 8 vo. Cloth, viii+374 pages. Price, $4.00. Cambridge University Press, Cambridge, England. G. P. Putnam's Sons, American Agents.

This treatise develops the theory of the dynamics of a particle of a rigid body, with that lucidity of expression, simplicity and directness of presentation, and logical consistency so characteristic of Professor Loney's Mathematical Texts. The book is intended for both engineering and mathematical students. A fair working knowledge of the Differential and Integral Calculus, as well as of Differential Equations, is required in order to read it, though all the Differential Equations required are solved in an appendix.

A large number of interesting problems, chiefly collected from University and College examinations, are included, and many problems are solved, thus giving the student a good idea how the various problems may be attacked.

The typography and mechanical execution of the work is all that could be desired. F.

Mechanics. By John Cox, M. A., F. R. S. C., Honorary LL. D., Queen's University, Kingston; formerly Professor of Physics in McGill University, Montreal; sometime Fellow of Trinity College, Cambridge, England. 8 vo. Cloth, xiv+332 pages. Price, $2.75. Cambridge: The University Press. G. P. Putnam's Sons, American Agents.

This book may be considered as a protest against the prevailing method of presenting the subject of mechanics, viz.: The method beginning the subject by collection of definitions of mass, motion, force, matter, etc.. comprising the first chapter and the second taking up the mathematical study of motion in the abstract, and so through a course which leaves the student with the notion that he has been studying mathematics pure but not simple. The author has fashioned his course of study somewhat after the famous treatise of Professor Ernst Mach. In writing the book the author has kept in view six aims, of which we mention the following:

(1) First and throughout, to make a text-book of mechanical principles, avoiding as far as possible merely mathematical difficulties; (2), to develop the principles in their historical order; (3), to bring out incidentally the points of philosophic interest and the method of science; and (4), to interest the student in the personality of the great pioneers of the science and if possible have them refer to their writings.

The course as presented in this volume has been tested by the author with his classes in McGill University, and has proved very satisfactory to him.

We believe that it would be well to have some such course as this to serve as an introduction to the study of mechanics, without regard to the future use the student wishes to make of the knowledge thus gained. The author believes that, "until mechanics is clad in its historical flesh and blood, it will remain the dull and tiresome subject that has convinced so many generations of students that an abysmal gulf separates theory from practice."

The text also contains portraits of Archimedes, Galileo, Huyghens, and Newton. F.